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# Age at death from a radiation-induced cancer based on the Marshall model for mortality period

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ABSTRACT

Results presented elsewhere in this issue of *Process Safety and Environmental Protection* point to the radiation-induced loss of life expectancy following severe nuclear accidents being lower than generally feared. But this leaves open the question of the loss of life expectancy amongst radiation cancer victims, even if fortunately there are likely to be few of them. Addressing this question, the research presented here finds that the average radiation cancer victim will live into his or her 60s or 70s, depending on how long the radiation exposure lasts, based on data from the UK life tables. Between 8 and 22 years of life expectancy will be lost, well below the 42 years taken away on average by an immediately fatal accident, such as a car crash or rail crash. Not only are the results useful in their own right, but they inevitably call into question once again the concept of the Value of a Prevented Fatality still used for cost-benefit analyses in the UK on a "one size fits all" basis, which disregards the amount of life expectancy lost. This problem with applying the VPF in the context of radiological protection is additional to the gross flaws previously uncovered in the value assigned to the VPF in the UK. It is clear that the VPF should not be used as a criterion for cost-benefit analysis in radiological protection.

An important feature of the results presented is that they apply to any exposure to radiation between a point dose and a constant annual dose that does not cause radiation sickness. The figures presented, for both point and constant annual exposures, are equally valid whether the dose is a few mSv or a few hundred mSv. Nor is the outcome affected by the magnitude of the coefficient used to convert radiation dose into risk.



## 1. Introduction

In his pioneering study, Lord Marshall (Marshall et al., 1983) calculated the loss of life expectancy after a severe nuclear accident and radioactivity release by assuming that any death caused by a radiation-induced cancer from a non-acute dose of radiation would occur at a random point between times $\omega_1$ and $\omega_2$ after exposure. He used a uniform distribution starting at $\omega_1 = 10$ and finishing $\omega_2 = 40$ years after exposure to represent the probability from death from any type of cancer caused by the radiation exposure. The possibility that the probability density for death from a radiation cancer will stay raised beyond 40 years is catered for conservatively in this formulation, in the sense that the Marshall model will predict a greater loss of life expectancy due to radiation exposure in such a circumstance. Any hazard that is delayed past 40 years is effectively brought forward.

Confirmatory evidence for the Marshall range, $\omega_1 = 10$ and $\omega_2 = 40$ years, comes from Chernobyl, where, while an excess of childhood thyroid cancer cases was detected as soon as 4 years after the 1986 accident (Thomas, 1997), the mean latency period before diagnosis was found to be 17 years with a standard deviation of 10 years (Thomas and Zwissler, 2003). Moreover, the figures are for first diagnosis rather than death, and, indeed, medical treatment will have produced full remission most cases (at least 70% of cases, Thomas and Zwissler, 2003, and possibly as many as 98%, UNSCEAR, 2000). Table A.4.2 of ICRP Publication 103 (ICRP, 2007c) contains five estimates in the range 90% to 93.4% for the fraction of people

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expected to survive a radiation-induced thyroid cancer. Clearly any deaths (affecting only a small fraction of the total cases diagnosed) would lag on incidence and diagnosis, but would likely fall within the Marshall time limits in most cases. But ICRP Publication 103 lists 14 different types of radiation-induced cancer in Table A.4.1, each of which will have different characteristics, including latency period. That those latency periods can be many decades long is shown by evidence from the Life Span Study of the Japanese survivors of the atomic bomb explosions (Preston et al., 2003, 2007). Marshall's choice of a uniform probability density between 10 and 40 years after exposure for death from radiation-induced cancer makes a reasonable allowance for the different latency periods associated with the large number of radiation-induced cancers that are possible. This is corroborated by a study made by Richardson and Ashmore (2005) of the records of 40,000 Canadian radiation workers (86% men, 14% women), who, since the 1950s, had received low level radiation doses. Cumulative whole body doses averaged 14 mSv, with the 90th percentile being 35 mSv. They found that for cancers other than lung and leukaemia there was a delay of between 5 and 15 years between exposure and the first radiation-induced mortalities and this was followed by a roughly uniform risk of death to 40 years and beyond. For lung cancer the delay was between 10 and 20 years, after which the risk of death was roughly uniform out to 30–35 years, when the risk fell to zero. They found for leukaemia that there was no lag on exposure (although they considered this unrealistically short) and the risk was constant out to 30–33 years, after which it dropped to zero.

The perturbations to the hazard rate that follow from Marshall's model are well suited to and have been adopted for use in *J*-value calculations (Thomas et al., 2006, 2010). Modifications to the coefficients have been introduced to account for the latest recommendations of the International Commission on Radiation Protection (ICRP, 1990, 2006, 2007a,b,c; Thomas and Jones, 2009a) and the method has been extended to take account of prolonged exposures as well as the one-off dose considered by Marshall (Thomas et al., 2006, 2007; Jones et al., 2007a,b; Jones and Thomas, 2009; Thomas and Jones, 2009b). As explained in Section 4.2 of Waddington et al. (2017), the calculated loss of life expectancy accords well with results from an independent computer code produced by the Centre d'étude sur l'Evaluation de la Protection dans le domaine Nucléaire (CEPN), as reported in Lochard and Schneider (1992). However, given that the values of $\omega_1$ and $\omega_2$ were chosen as representative (but conservative), the present work includes sensitivity studies, in the first of which $\omega_1$ was reduced to 5 years, while $\omega_2$ was increased to 45 years in the second.

The loss of life expectancy experienced across the population affected by radiation exposure is typically very small. For example, the members of the public most under threat from Chernobyl, namely the 116,000 people evacuated in the first phase, can be calculated to have lost 9.3 days of life expectancy as a result of the 1986 accident. This figure may be compared with the 3¼ year shortfall in population-average life expectancy experienced by the inhabitants of Manchester in the North of England when compared with that of the inhabitants of Harrow in London. It may be further compared with the estimated 4½ months by which Londoners' lives would on average be extended if the capital's air pollution could be reduced to rural levels (9 months at birth, Lord Darzi, 2014). Given that most people will prefer life to death, the population-average change in life expectancy provides a very good indicator of the size of harm, whether that harm results from industrial pollution or radiation pollution, for example. The average change in life expectancy over an appropriately defined population is the figure needed for a *J*-value analysis of measures to mitigate radiation exposure, where the *J*-value method of assessment has been validated against national (Thomas, 2017) and pan-national data (Thomas and Waddington, 2017a). However further useful information may be gained by calculating the loss of life expectancy for those who actually contract a radiation induced cancer that proves fatal.

The change in life expectancy will depend on the profile of the radiation dose with time. For simplicity this paper will consider that the dose is uniform over time, but will vary the time of exposure, $T_R$, to give an indication of the sensitivity of the results to exposure time. Thus $T_R$ may be set to 10 years, 20, 50, 100 or 10,000 years.

Meanwhile, choosing $T_R$ as 0.01 years (4 days) will give a good characterisation of the effects of a dose coming from a one-off release. This is verified using a separate approach applied to a point exposure. It will further be argued that these dose profiles, continuing exposure at a constant rate and a point exposure, bracket a range of other dose profiles, such as generated by fallout after a big nuclear accident, and give indicative results that are generally representative of those cases also.

The paper proposes a number of mathematical models for analysing the loss of life expectancy among those members of the public who will die as a result of exposure to nuclear radiation. Such unfortunate people, in practice a small fraction of those exposed even after a major accident, are defined as "radiation cancer victims".

The layout of the paper is as follows. Section 2 derives the conditional probability density function for contracting cancer at age, $x$, given that the radiation cancer victim's exposure started at age, $a$, named here his "starting age". This is then combined with a general form of the probability density for the mortality period, which is the difference, $m = y - x$, between the age at death, $y$, and the age, $x$, of cancer induction. The combination leads to the radiation cancer victim's probability density for death at age, $y$, given his starting age, $a$.

Section 3 specifies that the mortality period should obey the Marshall model, where the probability density for the mortality period is a rectangular function, uniform between time limits: $\omega_1$, which represents the minimum delay between cancer induction and death, and $\omega_2$, which represents the maximum time that death can be delayed on induction. The conditional probability density for death at age, $y$, given first exposure at age, $a$, is then derived for starting ages that are sufficiently low for all possible mortality periods to play out before the person dies of extreme old age. Extreme old age, $\alpha_0$, is set to 101 years in the calculations of this paper. Section 4 extends the treatment so that the conditional probability density for death at age, $y$, given that radiation exposure starts at age, $a$, can apply to all starting ages, without limit.

Section 5 uses the conditional probability density derived in Section 4 to find the expected age at death, $E(Y|A = a)$, for a radiation cancer victim of starting age, $a$, who is alive at the start of the exposure. The variance, $var(Y|A = a)$, is also found.

Section 6 introduces the steady state population model, where there is a turnover of people, but the overall number stays steady. Such a model can be used to represent to reasonable accuracy both national and urban populations. The section applies the steady-state model in deriving the probability density for starting age amongst radiation cancer victims who are alive at the start of the radiation exposure.

The probability density for starting age derived in Section 6 may be applied along with the expected age at death, $E(Y|A = a)$, for a radiation cancer victim of starting age, $a$, with the conjunction allowing the unconditional expected age at death for radiation cancer victims, $E(Y)$, to be found. This process is detailed in Section 7. The unconditional variance of age at death for radiation cancer victims, $var(Y)$, is also found in Section 7. These values apply to those alive at the time the radiation exposure starts.

But a prolonged radiation exposure will have an effect not only on those alive at the start of the exposure, but also on those born into it. It is now necessary to extend the treatment of Section 6 and find the probability density for starting age amongst radiation cancer victims for both those alive at the start of the exposure and those born into it. This is carried out in Section 8.

Section 9 uses this probability for starting age in an analogous fashion to Section 7, and the unconditional average age at death for radiation cancer victims, $E(Y)$, is found for the total population of radiation cancer victims, both those alive at the start of the exposure and those born into it. The corresponding unconditional variance of age at death for radiation cancer victims, $var(Y)$, is also found in Section 9 for the total population of radiation cancer victims subjected to a continuing radiation dose.

Section 10 treats the simpler problem of finding the average age at death for radiation cancer victims exposed to a point dose of radiation. $E(Y)$ and $var(Y)$ are derived. Since the point dose is the limiting case of a continuing dose as the dose period is reduced towards zero, the numerical results can be used as a check on the values produced under the continuing dose case.

Section 9's results for average age at death, $E(Y)$, and variance of the age at death, $var(Y)$, for radiation cancer victims do not require the probability density for age at death, $f_Y(y)$. But Section 11 uses previous intermediate results to derive this probability density, $f_Y(y)$. This has the advantage of not only allowing a further check on the results for $E(Y)$ and $var(Y)$but also allowing the cumulative probability of a radiation cancer victim living to a specified age to be found.

Results are presented in Section 12 under the headings:

- Radiation cancer victims of all starting ages
- One-off dose
- Radiation cancer victims with a starting age of zero
- Life expectancy of the radiation cancer victims
- Sensitivity Study 1: the delay before a radiation cancer death can be caused is reduced by 5 years: $\omega_1 = 5$
- Sensitivity Study 2: the delay after which the risk of dying from a radiation cancer disappears is increased by 5 years: $\omega_2 = 45$
- Applicability of the results to a large nuclear accident.

Section 13 contains a Discussion and Section 14 contains the Conclusions.

Appendix A contains an introduction to step or jump functions. Appendix B provides a mathematical characterisation of a steady state population and derives the probability distribution for age that is associated with it. Appendix C derives the properties of an important integral used in the main text. Appendix D lists the main assumptions and definitions, while Appendix E provides a nomenclature list.

## 2. Fundamental equations determining the probability density for death at age, *y*, for a radiation-cancer victim aged *a* at the start of the exposure

The ICRP assumption is used that, for each of the two broad classes into which a population may be divided, the general public and nuclear-industry workers, the probability of death from a radiation-induced cancer is proportional only to the dose of radiation received. The constant of proportionality is the same for members of the same class but is 25% greater for the general public, which will have members across the full age range: from very young people, infants, to very old people (ICRP, 1990, 2007a,b; Thomas and Jones, 2009a). The ICRP model makes no attempt to predict any possible interaction of radiation exposure with some other condition or habit, such as smoking, that might conceivably affect the probability of radiation harm. On the other hand, being based on population averages, the risk coefficients will contain some implicit allowance for the possible effects of such conditions in the observed populations that provide the data from which the coefficients are calculated.

For the specific purpose of this paper, a "radiation cancer victim" will be defined as an individual who will die prematurely as a result of contracting a cancer induced by exposure to ionising radiation at some previous time. Note that, by this definition, someone who contracts a radiation-induced cancer but does not die as a result of it, while undoubtedly an injured party, is not classed as a radiation cancer victim and may be called a radiation cancer survivor.

In line with the ICRP model, it is assumed that the probability of a fatal radiation induced cancer is independent of any possible stresses imposed by another deleterious condition, and that a radiation cancer victim will die solely as a result of his or her radiation exposure. This has the important implication that he or she will not suffer a premature death from any other cause such as an accident or a different fatal disease.

The words, "premature" and "prematurely", in the last paragraph imply the existence of some age, which we will denote $\alpha_0$, at which death would not be regarded as premature. We may regard $\alpha_0$ as the age of irreparable wear out, analogous to the universal break-down of Oliver Wendell Holmes's one-hoss shay (Encyclopaedia Britannica, 2015). Life tables in the UK cover ages up to and including 100 years, making it convenient to chose the age of death from extreme old age as $\alpha_0 = 101$, an approximation that will cover about 98% of the general population. People will usually die of one of a number of diverse causes before reaching this age, but, as discussed in the previous paragraph, the radiation-cancer victim will be immune to all such deadly hazards except for radiation-induced cancer.

Given the ICRP's provision of a single risk coefficient for members of the public (both genders and all ages) and following ICRP practice of assuming no lower threshold for radiation risk, the probability of persons of all ages and both genders contracting a fatal cancer is assumed to be proportional to the radiation dose received. Such a cancer will have a latency period before it is detected and death will follow some time later still.

Let a radiation exposure start at time, $t = 0$, and continue until $t = T_R$ years, where the subscript, $R$, may be taken to denote "radiation exposure". All the people living in the exposed community will experience the start of the radiation exposure at the same instant of time, although, in general, they will have a different age, $a$, at the start of their exposure.

An individual's age, $X$, may be referred to his age, $A$, at first exposure and the time, $T$, that has passed since the start of exposure, where $X$, $A$ and $T$ may be regarded as random variables. The exposure is assumed to begin at time, $t = 0$, and the individual's age, $X$, at a randomly chosen later time, $T$, will be

$$X = A + T \tag{1}$$

The person's age at first exposure may be described as his "starting age". Clearly the time, $T = t$, that has passed since the start of the exposure may be recovered from a person's current age, $X = x$, given his age at the start of the exposure, $A = a$:

$$t = x - a \tag{2}$$

Since the probability of death from a radiation-induced cancer is proportional to the dose of radiation received, the probability that a fatal cancer will be induced in a radiation cancer victim in any given interval will be equal to the ratio of the harmful

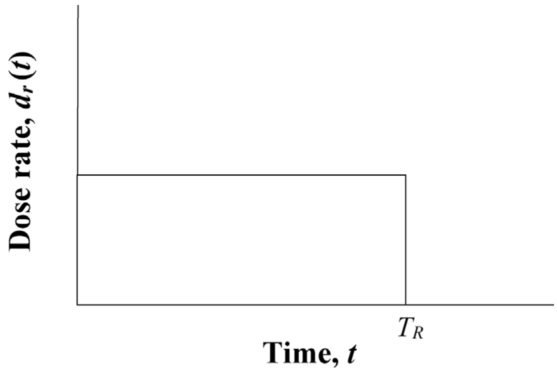


**Fig. 1 – Radiation dose, $d_r$ (t) vs. time, t.**

radiation received in that interval to the total dose of harmful radiation that the victim receives. Coding this in mathematics, the probability, $q_X(x)\,dx$, that the radiation-cancer victim will die at some later age as a result of an exposure occurring between times $t$ and $t+dt$ and so between ages $x$ $(=a+t)$ and $x+dx$ $(=a+t+dt)$ will be equal to the fraction received in that interval of the maximum integrated dose, $D_h$, that can harm him or her; the concept of the maximum harmful integrated dose and how it may be calculated are explained in the next paragraph. Hence

$$q_X(x)\,dx = \frac{d_{rx}(x)\,dx}{D_h} \tag{3}$$

in which $d_{rx}(x)$ is the dose rate experienced at age, x, and $q_X(x)$ is the probability density for a fatal radiation cancer being caused by exposure in the vicinity of time, t, and hence in the vicinity of age, $x=a+t$. Here the term in "the vicinity of z" for a general variable, z, is taken to mean between z and $z+dz$. The dose rate, $d_{rt}(t)$, as a function of time since the start of the exposure is related to $d_{rx}(x)$ by:

$$d_{rt}(x-a) = d_{rx}(x) \tag{4}$$

It may be noted that the definition of the maximum integrated dose, $D_h$, in the previous paragraph is subject to the qualifier, "harmful", which will now be clarified. Although the general premise used in this paper is that there is no threshold below which radiation dose can be regarded as harmless (the "linear, no threshold" (LNT) assumption), it is clear at the outset that no individual can receive a harmful radiation dose beyond the point when he is about to die anyway. Since the individual cannot by assumption live past the limiting old age, $\alpha_0$, common to all, it is clear that any otherwise harmful radiation that the individual's body receives after that age can have no effect. In fact, because of the delay in time, $\omega_1$, before harm from low-level radiation can be caused in the Marshall model, only radiation received before the individual reaches the age, $\alpha_0-\omega_1$, can have any harmful effect on him. Hence in considering an individual with starting age, $a$, we may restrict our attention to the ages, $x$, in the interval, $a<x\le\alpha_0-\omega_1$, for which the corresponding interval for time, t, is: $0<t\le\alpha_0-\omega_1-a$. Thus the maximum harmful dose that an individual with starting age, $a$, can receive is:

$$D_h(a) = \int_{x=a}^{\alpha_0-\omega_1} d_{rx}(x)\,dx = \int_{t=0}^{\alpha_0-\omega_1-a} d_{rt}(t)\,dt \tag{5}$$

For a radiation cancer victim, Eq. (3) may be integrated from his starting age, $a$, to the maximum age, $\alpha_0-\omega_1$, at which it is possible for low-level radiation to harm him. Thus:

$$\int_{x=a}^{\alpha_0-\omega_1} q_X(x)\,dx = \frac{1}{D_h(a)}\int_{x=a}^{\alpha_0-\omega_1} d_{rx}(x)\,dx = \frac{D_h(a)}{D_h(a)} = 1 \tag{6}$$

where Eq. (5) has been used in the last step. The fact that the right-hand side of Eq. (6) is unity confirms that the range of ages, x, for which $q_X(x)$ is valid as a probability density is $a\le x\le\alpha_0-\omega_1$. This range is suitable for use in analysing the effects on radiation-cancer victims, who are certain to contract a fatal radiation cancer at some age between $a$ and $\alpha_0-\omega_1$.

As is clear from Eq. (5), the maximum harmful radiation dose a person may receive, $D_h$ is age-dependent, *viz.* $D_h = D_h(a)$, which means that the probability density, $q_X(x)$, given by Eq. (3), for a fatal radiation cancer being caused by exposure at age, x, will depend on the starting age, $a$, so that we may write more accurately $q_{X|A}(x|a)$.

To proceed further, we need to specify the profile of the radiation exposure over time. We may usefully consider the case of a radiation exposure that starts at time 0 and remains constant at a dose of $d_a$ per year, over a period of $T_R$ years before ceasing, as shown graphically in Fig. 1. Setting $T_R$ at, say, 10 years, this profile can be used represent a prolonged exposure, a decade long in this instance, such as might be associated with continuing, low-level releases from a nuclear plant. On the other hand, putting $T_R = 0.01$ will provide a good approximation to a one-off exposure at a single point in time. We shall apply the model to the general public, who will have a full spread of ages from 0 to $\alpha_0$.

The dose rate, $d_{rt}$, as a function of time, $t$, may be represented using a step or "jump" function, $J_p(z)$:

$$d_{rt}(t) = d_a(1 - J_p(t - T_R)) \tag{7}$$

where

$$J_p(z) \begin{aligned} &= 0 \ \text{ for } \ z < 0 \\ &= 1 \ \text{ for } \ z \geq 0 \end{aligned} \tag{8}$$

The dose rate in terms of age, $x$, for an individual of starting age, $a$, is therefore:

$$d_{rx}(x) = d_{rt}(x - a) = d_a(1 - J_p(x - a - T_R)) \tag{9}$$

where use has been made of Eq. (4) in the first step and Eq. (7) in the second. Eq. (7) may, of course, be recovered immediately from Eq. (9) by using Eq. (2).

The total dose that a notional person could receive if exposed over the whole period, $T_R$ (or series of single persons in the case of a very long exposure) is given by:

$$D_r = \int_{t=0}^{\infty} d_{rt}(t)\,dt = \int_{t=0}^{\infty} d_a(1 - J_p(t - T_R))\,dt = d_a T_R \tag{10}$$

However the total harmful dose that an individual of age, $a$, will receive is not $D_r$ but $D_h(a)$, given by:

$$D_h(a) = \int_{x=a}^{\alpha_0-\omega_1} d_{rx}(x)\,dx = \int_{t=0}^{\alpha_0-a-\omega_1} d_{rt}(t)\,dt \tag{11}$$

Hence

$$\begin{aligned} D_h(a) &= d_a \int_{t=0}^{\alpha_0-a-\omega_1} (1 - J_p(t - T_R))\,dt \\ &= d_a(\alpha_0 - a - \omega_1 - (\alpha_0 - a - \omega_1 - T_R)J_p(\alpha_0 - a - \omega_1 - T_R)) \\ &= d_a T_R J_p(\alpha_0 - a - \omega_1 - T_R) \end{aligned} \tag{12}$$

where the integration of the jump function, namely $\int_{t=0}^{\alpha_0-a-\omega} J_p(t - T_R)\,dt$, uses Eq. (A.1) from Appendix A, which lays out a brief calculus of step or jump functions.

Substituting from Eqs. (9) and (12) into Eq. (3) produces the probability density, $q_{X|A}(x|a)$, for death from cancer as a result of radiation experienced at age, $x$, given that the starting age is $a$:

$$\begin{aligned} q_{X|A}(x|a) &= \frac{d_{rx}(x)}{D_h} = \frac{d_a(1 - J_p(x - a - T_R))}{d_a T_R J_p(\alpha_0 - a - \omega_1 - T_R)} \\ &= \frac{1 - J_p(x - a - T_R)}{T_R J_p(\alpha_0 - a - \omega_1 - T_R)} \end{aligned} \quad 0 \leq a \leq \alpha_0 - \omega_1 - T_R;\, T_R > 0 \tag{13}$$

The condition, $a \leq \alpha_0 - \omega_1 - T_R$, is necessary because the jump function in the denominator is zero for $a > \alpha_0 - \omega_1 - T_R$, leaving the probability density, $q_{X|A}(x|a)$, improperly defined. A point radiation exposure may be modelled simply by shrinking the period of radiation exposure to a very low value of $T_R$, e.g. $T_R = 0.01$ years. An alternative treatment for a one-off dose is given in Section 10.

Confining our attention to the computable region, $0 \leq a \leq \alpha_0 - \omega_1 - T_R$, we may simplify Eq. (13) to:

$$q_{X|A}(x|a) = \frac{1 - J_p(x - a - T_R)}{T_R} \quad 0 \leq a \leq \alpha_0 - \omega_1 - T_R \tag{14}$$

Consider an individual who will die as a result of a radiation-induced cancer, by our definition a radiation cancer victim. When he is subject to a continuing dose, his cancer will be induced when he has a random age, $X$, which will be at or above his "starting age", $A$, his age when the exposure starts. Thus $X \geq A$. He will die after a further interval of time, the random mortality period, $M$, when he has a random age, $Y$, where:

$$Y = M + X \tag{15}$$

The random variables, $X$ and $M$, are independent, meaning that the length of the mortality period, $M$, does not depend on the age at induction, $X$. This is a feature of the Marshall model for mortality period, which is discussed in Section 3 and which

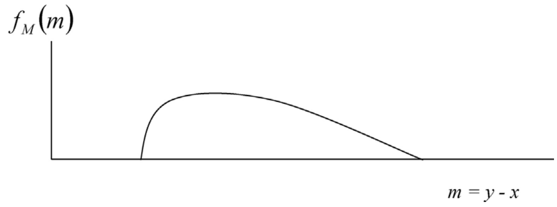


**Fig. 2 – Generalised probability density for the mortality period, *m*.**

constitutes a starting point for the developments in this paper. Marshall's assumption is backed up by Richardson and Ashmore (2005), who, in their study of 40,000 Canadian radiation workers, 86% male, 14% female, found that there was

"minimal evidence of variation in exposure effects [with age] for cancers other than lung and leukaemia".

They discovered that the risk of dying from a radiation-induced cancer other than lung cancer and leukaemia was the same irrespective of the age at exposure. For lung cancer and leukaemia they found that the vulnerability to a radiation-induced cancer climbed from zero at age 15 to a constant level for ages at exposure above $33 \pm 2$ years.

Let $f_M(m)$ be the probability distribution for the random mortality period, $M$. A generalised representation of such a probability density is given in Fig. 2. The probability for death in the vicinity of some future age, $y$, following induction in the vicinity of age, $x$, where $x \geq a$, is simply the product of the probability of induction between ages, $x$ and $x + dx$, already introduced as $q_{X|A}(x|a)\,dx$, and the probability that the mortality period lies between $m$ and $m + dm$, namely $f_M(m)\,dm$. This product is $f_M(m)\,q_{X|A}(x|a)\,dxdm$. Since the mortality period is the difference between the age at death, $y$, and the age, $x$, of cancer induction, we may write $m = y - x$ and also conclude that $dm = dy$, for a given age of exposure, $x$. Hence we may write the equation between probabilities: $f_M(m)\,q_{X|A}(x|a)\,dxdm = f_M(y - x)\,q_{X|A}(x|a)\,dxdy$.

The last term represents the probability density for the death at age, $y$, for a radiation-cancer victim for whom induction occurred at age, $x$, where $a$ is the victim's age at the start of the radiation exposure. But death at age, $y$, could have been caused by a cancer induced over a large range of possible, earlier ages, $x$. To find the total probability density for death from radiation-induced cancer at age, $y$, for an individual with starting age, $a$, we need to integrate from the age, $x = a$, at the start of the radiation exposure to the chosen age, $y$:

$$f_{Y|A}(y|a) = \int_{x=a}^{y} f_M(y - x)\,q_{X|A}(x|a)\,dx \tag{16}$$

It is appropriate to note at this point that the expression, $f_{Y|A}(y|a)$, depends, via $q_{X|A}(x|a)$ in Eq. (13), on the ratio of a dose rate to the total integrated dose that can cause harm to the individual of a given age. No reference has been made to the size of the ICRP risk coefficient, only to that organisation's assumptions of no lower threshold and a common figure for all members of the public. Nor has any dose rate been specified, beyond the fact that it is not high enough to cause acute effects. Hence the expression will apply for constant dose rates either below or above the point (100 mSv per year) where an increase in risk coefficient is recommended. Furthermore, it is proof against any possible changes that the ICRP might make to the estimated risk coefficient in the future.

## 3. Applying the Marshall model for mortality period

By the Marshall model, the probability density for mortality period is uniform over the interval of length, $\Omega$ years, between ages $a + \omega_1$ and $a + \omega_2$, where:

$$\Omega = \omega_2 - \omega_1 \tag{17}$$

Hence

$$f_M(m) = \begin{cases} 0 & \text{for} \quad 0 < m \leq \omega_1 \\ \dfrac{1}{\Omega} & \text{for} \quad \omega_1 < m \leq \omega_2 \\ 0 & \text{for} \quad m > \omega_2 \end{cases} \tag{18}$$

See Fig. 3. Justification for this simplified mathematical model is provided in the Introduction to this paper.

Using jump functions, equation set (18) may be re-expressed as the single equation:

$$f_M(m) = \frac{J_p(m - \omega_1) - J_p(m - \omega_2)}{\Omega} \tag{19}$$

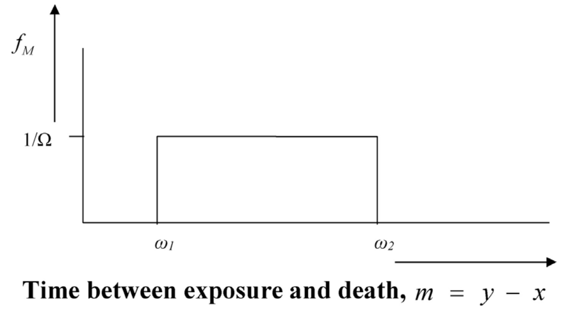


**Fig. 3 – The Marshall probability density for death from a radiation induced cancer at an age, y, after an exposure at age, x.**

Returning now to Eq. (16), we may substitute from Eqs. (13) and (19) to give:

$$f_{Y|A}(y|a) = \frac{1}{\Omega T_R}\int_{x=a}^{y} (J_p(y-x-\omega_1) - J_p(y-x-\omega_2))(1-J_p(x-a-T_R))\,dx \quad \text{for } 0 \le a \le \alpha_0 - \omega_1 - T_R \tag{20}$$

This integral with respect to age, x, may be transformed into a formulation in terms of time, t, through the use of Eq. (2), differentiation of which gives $dx = dt$. The same Eq. (2) shows that $t = 0$ when $x = a$, while when $x = y$, then $t = y - a$. Now let us denote by $\tau$ the difference between the age at death, y, for a radiation cancer victim and his age, $a$, at the start of exposure:

$$\tau = y - a \tag{21}$$

This allows the integral in Eq. (20) to be expanded as:

$$\begin{aligned}\int_{x=a}^{y} (J_p(y-x-\omega_1) - J_p(y-x-\omega_2))(1-J_p(x-a-T_R))\,dx &= \int_{t=0}^{y-a} (J_p(y-(a+t)-\omega_1) - J_p(y-(a+t)-\omega_2)) \\ \times (1-J_p((a+t)-a-T_R))\,dt &= \int_{t=0}^{\tau} (J_p(\tau-t-\omega_1) - J_p(\tau-t-\omega_2))(1-J_p(t-T_R))\,dt\end{aligned} \tag{22}$$

Let us define the function, $\psi_0(\tau)$, as the integral:

$$\psi_0(\tau) = \int_{t=0}^{\tau} (J_p(\tau-t-\omega_1) - J_p(\tau-t-\omega_2))(1-J_p(t-T_R))\,dt \tag{23}$$

Physical reasoning allows an analytic solution of this integral as:

$$\psi_0(\tau) = (\tau-\omega_1)J_p(\tau-\omega_1) - (\tau-\omega_2)J_p(\tau-\omega_2) - (\tau-\omega_3)J_p(\tau-\omega_3) + (\tau-\omega_4)J_p(\tau-\omega_4) \tag{24}$$

where

$$\omega_3 = T_R + \omega_1 \tag{25}$$

$$\omega_4 = T_R + \omega_2 \tag{26}$$

This result, which may be confirmed by numerical integration of Eq. (23), is plotted in Fig. 4 for a number of values of $T_R$: $1 \le T_R \le 50$ years. Putting $a = 0$ in Eq. (20) produces $\tau = y$, which allows Fig. 4 to be interpreted as the function, $\psi_0(y)$, plotted against age at death, y, for someone of starting age, $a = 0$.

Substituting from Eqs. (21)–(24) into Eq. (20), the probability density for death at age, y, for a radiation victim of starting age, $a$, is

$$f^*_{Y|A}(y|a) = \frac{\psi_0(\tau)}{\Omega T_R} = \frac{\psi_0(y-a)}{\Omega T_R} \tag{27}$$

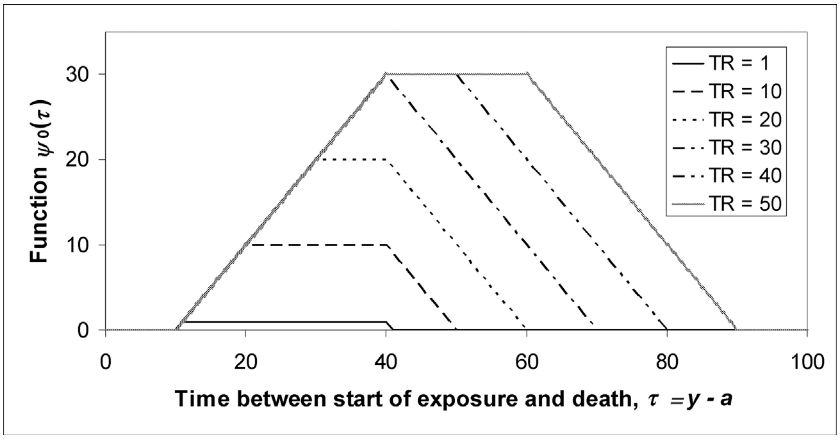


**Fig. 4 – Function $\psi_0(\tau)$ versus the time, $\tau = y - a$, between start of exposure and death for an individual of starting age, $a$.**

An asterisk has been applied in $f^*_{Y|A}(y|a)$ to signify that the range of starting ages, $a$, for which the probability density is valid is limited. We know that the condition imposed on $q_{X|A}(x|a)$ in Eq. (14) will carry forward to the probability density, $f^*_{Y|A}(y|a)$, a condition that may be written

$$0 \le a \le \alpha_0 - \omega_3 \tag{28}$$

after applying Eq. (25). But in fact a stricter upper limit on starting age, $a$, needs to be imposed on Eq. (27). For $f^*_{Y|A}(y|a)$ to qualify as a probability density, it is necessary that for any starting ages, $a: 0 \le a \le \infty$, $\int_{\text{all feasible } y} f^*_{Y|A}(y|a)\,dy = \int_{y=a}^{\infty} f^*_{Y|A}(y|a)\,dy = 1$. Integrating Eq. (27) gives:

$$\int_{y=a}^{\infty} f^*_{Y|A}(y|a)\,dy = \frac{1}{\Omega T_R}\int_{y=a}^{\infty} \psi_0(y-a)\,dy = \frac{1}{\Omega T_R}\int_{\tau=0}^{\infty} \psi_0(\tau)\,d\tau \tag{29}$$

after the substitution from Eq. (21), $\tau = y - a$, is made. Moreover

$$\frac{1}{\Omega T_R}\int_{\tau=0}^{\infty} \psi_0(\tau)\,d\tau = \frac{1}{\Omega T_R}\int_{\tau=0}^{\omega_4} \psi_0(\tau)\,d\tau \tag{30}$$

since $\psi_0(\tau) = 0$ for $\tau \ge \omega_4$, as illustrated in Fig. 4.

Using Appendix A, Eq. (A.3), to introduce $\psi_1(z) = \int_{m=0}^{z} \psi_0(m)\,dm$, Eq. (30) may be rewritten:

$$\begin{aligned}\psi_1(\omega_4) &= \int_{\tau=0}^{\omega_4} \psi_0(\tau)\,d\tau \\ &= \frac{1}{2}\left\{(\omega_4-\omega_1)^2 J_p(\omega_4-\omega_1) - (\omega_4-\omega_2)^2 J_p(\omega_4-\omega_2) - (\omega_4-\omega_3)^2 J_p(\omega_4-\omega_3) + (\omega_4-\omega_4)^2 J_p(\omega_4-\omega_4)\right\} \\ &= \frac{1}{2}\left\{(\omega_4-\omega_1)^2 - (\omega_4-\omega_2)^2 - (\omega_4-\omega_3)^2\right\}\end{aligned} \tag{31}$$

where the last line uses the fact that $\omega_4 > \omega_3 > \omega_2 > \omega_1$. It may be noted that the function, $\psi_1(\omega_4)$, depends also on the value of the radiation exposure period, $T_R$, via Eqs. (25) and (26). This point could be emphasised by extending the notation to put $\psi_1(\omega_4) = \psi_1^{(T_R)}(\omega_4)$, a form that will be needed in Section 8, when the effective radiation exposure period for some people will be reduced below $T_R$.

Substituting from Eqs. (17), (25) and (26) into Eq. (31) gives:

$$\begin{aligned}\psi_1(\omega_4) &= \frac{1}{2}\left\{(\omega_1+\Omega+T_R-\omega_1)^2 - (\omega_1+\Omega+T_R-\omega_1-\Omega)^2 - (\omega_1+\Omega+T_R-\omega_1-T_R)^2\right\} = \frac{1}{2}\left\{(\Omega+T_R)^2 - T_R^2 - \Omega^2\right\} \\ &= \frac{1}{2}\left\{\Omega^2 + 2\Omega T_R + T_R^2 - T_R^2 - \Omega^2\right\} = \Omega T_R\end{aligned} \tag{32}$$

Applying this result to Eqs. (29) and (30) gives:

$$\int_{y=a}^{\infty} f^{*}_{Y|A}(y|a)\,dy = \frac{\Omega T_R}{\Omega T_R} = 1 \tag{33}$$

as required for a conditional probability density where $y \ge a \ge 0$. Eq. (33) expresses the truism that a radiation cancer victim needs to die of radiation cancer at some age.

## 4. Probability density for age at death, y, for victims of all starting ages, *a*

Eq. (30) will apply only when the integration upper limit, $\tau = \omega_4$, may be reached, which, since $\tau = y - a$ implies that the starting age, $a$, must lie at or below $y - \omega_4$. But $y = \alpha_0$ is the greatest age at which a person can be considered to have died from radiation cancer, since at this point death from extreme old age is assumed to intervene. Hence the form of probability density, $f^{*}_{Y|A}(y|a)$ as defined by Eq. (27) will be valid only for starting ages, $a$, in the range $0 \le a \le \alpha_0 - \omega_4$ or, making use of Eqs. (17) and (26):

$$0 \le a \le \alpha_0 - \omega_1 - T_R - \Omega \tag{34}$$

Interpreting this result graphically and referring to Fig. 4 it is necessary for the whole of the curve, $\psi_0(\tau)$, to play out to the point where $\psi_0(\tau)$ returns to zero, a circumstance that will occur at $\tau = \omega_4$ or $y = a + \omega_4$. Since $\omega_4 > \omega_3$, condition (34) on starting age, equivalent to $a \le \alpha_0 - \omega_4$, is clearly stricter than condition (28), namely $a \le \alpha_0 - \omega_3$.

However, condition (34) may be relaxed by generalising the form of Eq. (27) to a corrected form:

$$f_{Y|A}(y|a) = \begin{cases} \dfrac{\psi_0(\tau)}{\displaystyle\int_{\tau=0}^{\tau_{max}(a)} \psi_0(\tau)\,d\tau} & \text{for} \quad \tau \le \tau_{max}(a) \\ 0 & \text{for} \quad \tau > \tau_{max}(a) \end{cases} \tag{35}$$

where

$$\left.\begin{array}{l} \tau = y - a \\ \Rightarrow \tau_{max}(a) = \alpha_0 - a \end{array}\right\} \tag{36}$$

It is clear that the necessary condition, $\displaystyle\int_a^{\infty} f_{Y|A}(y|a)\,dy = 1$, is satisfied since, using Eqs. (29) and then (35), we may write

$$\int_{y=a}^{\infty} f_{Y|A}(y|a)\,dy = \int_{\tau=0}^{\infty} f_T(\tau)\,d\tau = \frac{\displaystyle\int_{\tau=0}^{\infty} \psi_0(\tau)\,dt}{\displaystyle\int_{\tau=0}^{\tau_{max}(0)} \psi_0(\tau)\,dt} = \frac{\displaystyle\int_{\tau=0}^{\tau_{max}(0)} \psi_0(\tau)\,dt}{\displaystyle\int_{\tau=0}^{\tau_{max}(0)} \psi_0(\tau)\,dt} = 1 \tag{37}$$

The minimum mortality period is $\omega_1$, meaning that the maximum value of starting age for someone subsequently to become a radiation cancer victim is $\alpha_0 - \omega_1$. Hence, after using Eq. (A.3) from Appendix A, the probability density for death at age, $y$, for a radiation cancer victim of starting age, $a$, is:

$$f_{Y|A}(y|a) = \left\{ \begin{array}{ll} \dfrac{\psi_0(\tau)}{\psi_1(\tau_{max}(a))} & \text{for} \quad \tau \le \tau_{max}(a) \\ 0 & \text{for} \quad \tau > \tau_{max}(a) \end{array} \right\} \quad \text{for} \quad a \le \alpha_0 - \omega_1 \tag{38}$$

## 5. The expected age at death, $E(Y|A=a)$, for a radiation cancer victim of starting age, $a$, living at the start of the exposure

The expected age at death, $E(Y|A=a)$, for a radiation cancer victim of starting age, $a$, may now be calculated as:

$$E(Y|A=a) = \int_{y=a}^{\alpha_0} y f_{Y|A}(y|a)\,dy = \frac{1}{\psi_1(\alpha_0-a)} \int_{y=a}^{\alpha_0} y\psi_0(y-a)\,dy = \frac{1}{\psi_1(\alpha_0-a)} \int_{\tau=0}^{\alpha_0-a} (\tau+a)\,\psi_0(\tau)\,d\tau \tag{39}$$

after using the substitution of Eq. (21), namely $\tau = y - a$. The integral term may be expanded further:

$$\int_{\tau=0}^{\alpha_0-a} (a+\tau)\,\psi_0(\tau)\,d\tau = a\int_{\tau=0}^{\alpha_0-a} \psi_0(\tau)\,d\tau + \int_{\tau=0}^{\alpha_0-a} \tau\psi_0(\tau)\,d\tau = a\psi_1(\alpha_0-a) + \theta_1(\alpha_0-a) \tag{40}$$

where expressions for $\psi_1(z) = \int_{m=0}^{z} \psi_0(m)\,dm$ and $\theta_1(z) = \int_{m=0}^{z} m\psi_0(m)\,dm$ are given in Appendix A, Eqs. (A.3) and (A.5) respectively.

Substituting from Eq. (40) into Eq. (39) gives the expected age at death for a radiation cancer victim of starting age, $a$, as

$$E(Y|A=a) = \frac{a\psi_1(\alpha_0-a) + \theta_1(\alpha_0-a)}{\psi_1(\alpha_0-a)} = a + \frac{\theta_1(\alpha_0-a)}{\psi_1(\alpha_0-a)} \tag{41}$$

Meanwhile the variance of the age at death for a radiation cancer victim of starting age, $a$, is given by

$$var(Y|A=a) = E\left(Y^2|A=a\right) - [E(Y|A=a)]^2 = \int_{y=a}^{\alpha_0} y^2 f_{Y|A}(y|a)\,dy - [E(Y|A=a)]^2 \tag{42}$$

The integral term in Eq. (42) may be re-expressed using Eq. (38) as:

$$\int_{y=a}^{\alpha_0} y^2 f_{Y|A}(y|a)\,dy = \frac{1}{\psi_1(\alpha_0-a)} \int_{y=a}^{\alpha_0} y^2\psi_0(y-a)\,dy = \frac{1}{\psi_1(\alpha_0-a)} \int_{\tau=0}^{\alpha_0-a} (a+\tau)^2\psi_0(\tau)\,d\tau \tag{43}$$

using the substitution, $\tau = y - a$. The integral term on the right-hand side of Eq. (43) may be expanded:

$$\int_{\tau=0}^{\alpha_0-a} (a+\tau)^2\psi_0(\tau)\,d\tau = a^2\int_{\tau=0}^{\alpha_0-a} \psi_0(\tau)\,d\tau + 2a\int_{\tau=0}^{\alpha_0-a} \tau\psi_0(\tau)\,d\tau + \int_{\tau=0}^{\alpha_0-a} \tau^2\psi_0(\tau)\,d\tau = a^2\psi_1(\alpha_0-a) + 2a\theta_1(\alpha_0-a) + \theta_2(\alpha_0-a) \tag{44}$$

where the additional function, $\theta_2(z) = \int_{m=0}^{z} m^2\psi_0(m)\,dm$, is given by Eq. (A.7) in Appendix A. Thus the variance in the age at death for a radiation cancer victim with starting age, $a$, is

$$var(Y|A=a) = a^2 + 2a\frac{\theta_1(\alpha_0-a)}{\psi_1(\alpha_0-a)} + \frac{\theta_2(\alpha_0-a)}{\psi_1(\alpha_0-a)} - [E(Y|A=a)]^2 \tag{45}$$

Substituting from Eq. (41) allows Eq. (45) to be written:

$$var(Y|A=a) = a^2 + 2a\frac{\theta_1(\alpha_0-a)}{\psi_1(\alpha_0-a)} + \frac{\theta_2(\alpha_0-a)}{\psi_1(\alpha_0-a)} - a^2 - 2a\frac{\theta_1(\alpha_0-a)}{\psi_1(\alpha_0-a)} - \left(\frac{\theta_1(\alpha_0-a)}{\psi_1(\alpha_0-a)}\right)^2 = \frac{\theta_2(\alpha_0-a)}{\psi_1(\alpha_0-a)} - \left(\frac{\theta_1(\alpha_0-a)}{\psi_1(\alpha_0-a)}\right)^2 \tag{46}$$

## 6. Probability density for starting age amongst radiation cancer victims living at the start of the exposure

It is now necessary to consider a population that is continually being replaced over time as people are born into it on an ongoing basis and from which people depart when they die, a process that also occurs on a continuing basis. If it is assumed that equal numbers are being born and dying each year, the population will remain constant over time. Such a steady-state idealisation brings mathematical simplification while retaining approximate validity for the populations of many developed countries. It may be applied to population subsets also, such as a town or large village, to which the actuarial statistics of the containing nation may be taken to be apply. The mathematics associated with the assumption of a steady-state population are described in Appendix B.

For a large population of $N$ individuals, the number, $N(a)$, of people of age, $a$, will be:

$$N(a) = \int_{a}^{a+1} N f_A(a)\, da \approx N f_A(a) \tag{47}$$

where the approximation follows from assuming that the probability density for age, $f_A(a)$, stays constant over the interval $a$ to $a+1$. Assuming a steady state population, $f_A(a)$, is given by Eq. (B.6) of Appendix B.

Suppose that the members of a public population with ages up to some value, $a_{max}$, are subjected to a continuing annual dose of radiation, $d_a$ (Sv/y), that starts at the same time for all and continues for a period of time, $T_R$. If no upper limit, $\alpha_0$, were imposed on lifetime for this group, then the members of each age group would experience the same dose over time and also the full, delayed effects of that dose. Given a risk coefficient with the general properties promulgated by the ICRP, viz. a common figure for all members of the public and no lower threshold for radiation risk, the fraction of radiation cancer victims in each age group, $c_v(a)$ would then take the same value, $c_v$, at all ages:

$$c_v(a) = c_v \quad \text{for} \quad \text{all } a \tag{48}$$

Under this preliminary scenario, the expected number of radiation cancer victims with starting age, $a$, would then be:

$$N_{v0}(a) \approx c_v N f_A(a) \tag{49}$$

Imposing no upper limit, $\alpha_0$, on lifetime or, equivalently, letting $\alpha_0 \to \infty$, allows the probability density for age at death, $y$, for a radiation cancer victim of starting age, $a$, to be given by Eq. (27) without restriction:

$$f_{Y|A}(y|a) = \frac{\psi_0(y-a)}{\Omega T_R} \tag{50}$$

Eq. (50) may be integrated with respect to age to give the probability of death before some specified value of age, $\beta_0$, as:

$$P(Y \le \beta_0|a) = \int_{y=a}^{\beta_0} f_{Y|A}(y|a)\, dy = \frac{1}{\Omega T_R}\int_{y=a}^{\beta_0} \psi_0(y-a)\, dy = \frac{1}{\Omega T_R}\int_{\tau=0}^{\beta_0-a} \psi_0(\tau)\, d\tau = \frac{1}{\Omega T_R}\psi_1(\beta_0-a) \tag{51}$$

where $\psi_1(z)$ is given by Eq. (A.3) in Appendix A. In fact, a more realistic model, and the one adopted in this paper, will impose some limit, $\alpha_0$, on age, so that a radiation cancer victim will see no effects after this point as death from extreme old age will have intervened. Only the fraction, $P(Y \le \alpha_0|a)$, of potential victims, $N_{v0}(a)$, dying from radiation induced cancer before age, $\alpha_0$, can be accounted actual radiation cancer victims. Hence the actual number of radiation cancer victims, $N_v(a)$, in each 1-year age band will be given by a modified version of Eq. (49), namely:

$$N_v(a) = N_{v0}(a)\, P(Y \le \alpha_0|a) = \frac{c_v N}{\Omega T_R}\psi_1(\alpha_0 - a) f_A(a) \tag{52}$$

The total number, $N_v$, of radiation cancer victims over all possible starting ages is then found by integrating:

$$N_v = \int_{a=0}^{\alpha_0} N_v(a)\, da = \frac{c_v N}{\Omega T_R}\int_{a=0}^{\alpha_0} \psi_1(\alpha_0 - a) f_A(a)\, da \tag{53}$$

Thus the probability density for starting age, $f_{vA}(a)$, amongst radiation cancer victims is given by:

$$f_{vA}(a) \approx \frac{N_v(a)}{N_v} = \frac{\psi_1(\alpha_0 - a)}{\int_{a=0}^{\alpha_0} \psi_1(\alpha_0 - a) f_A(a)\, da} f_A(a) \tag{54}$$

The calculation may be clarified by noting that $\psi_1(\alpha_0 - a) = 0$ for $a \ge \alpha_0 - \omega_1$, so that Eq. (54) may be rewritten:

$$f_{vA}(a) \approx \begin{cases} \dfrac{\psi_1(\alpha_0 - a)}{\int_{a=0}^{\alpha_0-\omega_1} \psi_1(\alpha_0 - a) f_A(a)\, da} f_A(a) & \text{for} \quad a < \alpha_0 - \omega_1 \\ 0 & \text{for} \quad a \ge \alpha_0 - \omega_1 \end{cases} \tag{55}$$

reflecting the fact that no ill effects can be felt from a radiation dose experienced by someone with a starting age at or above $\alpha_0 - \omega_1$.

## 7. Average age at death across all radiation-cancer victims living at the start of the exposure

Its dependence on the random variable, $A$, means that $E(Y|A)$ may be regarded as a random variable. Hence we find its expectation, which will be the expected value of age at death for all radiation cancer victims alive at the start of the radiation exposure:

$$E(Y) = E(E(Y|A)) = \int_{a=0}^{\alpha_0-\omega_1} f_{\upsilon A}(a)\left(a + \frac{\theta_1(\alpha_0 - a)}{\psi_1(\alpha_0 - a)}\right) da = E(A) + E\left(\frac{\theta_1(\alpha_0 - A)}{\psi_1(\alpha_0 - A)}\right) \quad (56)$$

Here $E(A)$ is the average starting age of radiation cancer victims, who will have ages up to $\alpha_0 - \omega_1$:

$$E(A) = \int_{a=0}^{\alpha_0-\omega_1} f_{\upsilon A}(a)\, a da \quad (57)$$

while the second term, $E(\theta_1(\alpha_0 - A)/\psi_1(\alpha_0 - A))$ represents an amount that is added on to the average starting age in order to find the average age at death:

$$E\left(\frac{\theta_1(\alpha_0 - A)}{\psi_1(\alpha_0 - A)}\right) = \int_{a=0}^{\alpha_0-\omega_1} f_{\upsilon A}(a)\frac{\theta_1(\alpha_0 - a)}{\psi_1(\alpha_0 - a)} da \quad (58)$$

We may also find the variance of $E(Y|A)$, which is the variance of the expected value of age at death for all radiation cancer victims alive at the start of the radiation exposure and will be given by:

$$var(E(Y|A)) = E\left([E(Y|A)]^2\right) - [E(E(Y|A))]^2 = \int_{a=0}^{\alpha_0-\omega_1} f_{\upsilon A}(a)(E(Y|A = a))^2 da - [E(Y)]^2 \quad (59)$$

where Eq. (56) has been used in the second step.

Meanwhile the dependence of $var(Y|A)$ on the random variable, $A$, means that $var(Y|A)$ may be regarded as a random variable also. Applying the expectation operator to Eq. (46) gives:

$$E(var(Y|A)) = \int_{a=0}^{\alpha_0-\omega_1} f_{\upsilon A}(a)\left(\frac{\theta_2(\alpha_0 - a)}{\psi_1(\alpha_0 - a)} - \left(\frac{\theta_1(\alpha_0 - a)}{\psi_1(\alpha_0 - a)}\right)^2\right) da \quad (60)$$

Finally, we may use the conditional variance formula (see e.g. Ross, 2003) to find the variance of the age at death for a radiation cancer victim as:

$$var(Y) = var(E(Y|A)) + E(var(Y|A)) \quad (61)$$

Eq. (61) may be computed using the results from Eqs. (59) and (60).

## 8. Probability density for starting age amongst radiation-cancer victims, both those living at the start of the exposure and those born into it

Finding the probability density for starting age amongst radiation-cancer victims, both those living at the start of the exposure and those born into it, requires an assumption about the dynamic behaviour of the overall population. An assumption of a steady-state population is used here.

The steady state assumption is a good approximation, for example, for a community living close to a nuclear installation discharging low-level nuclear effluent, where it is assumed that older people will die and younger people will be born into the community to replace them. By the same token, it will also be appropriate to assess the expected effects on a community subjected to a higher continuing radiation exposure, perhaps after a major nuclear release, provided the community remains in situ. It might be that the people in that settlement might actually move away to a new settlement if such an exposure were to occur. For example, 116,000 people were moved away from Chernobyl in the first relocation of 1986. Based on contamination measurements and dose rate models, it has been possible to calculate the loss of life expectancy that they would have experienced if they had not been evacuated based on the assumption that their communities were stable (in a steady state); see Waddington et al. (2017). Further estimates of the lower dose truly received allow a calculation of the loss of life expectancy the 116,000 relocated people will actually have experienced. In the case where the dose to those relocated has been delivered in the short time before being moved, the dose may be regarded as a point exposure and the relevant probability distribution for age is that which pertained just before moving, The life expectancy of the cohort can be calculated in such a case, and it does not matter whether or not the individuals remain together in a sustainable, new community.

The Marshall model calculates the additional hazard from radiation and adds this to the actuarial hazard rate, the other components of which represent the multiple and diverse hazards to which the person of a given age is naturally subject in the country in question. These other hazards, which are listed in summary numerical form in the life tables for the nation concerned (see, for example, Chiang, 1984), are assumed to dominate, so that the radiation hazard represents a small perturbation to the system. This assumption that the perturbation to the natural hazard rate will be small has been confirmed by the experience after Chernobyl, where childhood thyroid cancer has been the only cancer for which a possible increase after the accident has been large enough to detect (Thomas and Zwissler, 2003). Here the full remission rate is likely to be above 90%, as noted in the Introduction, so that the effect on mortality will be very small. Meanwhile the World Health Organisation (2006) provides this summary:

> "Projections concerning cancer deaths among the five million residents of areas with radioactive caesium deposition of 37 kBq/m$^2$ in Belarus, the Russian Federation and Ukraine are much less certain because they are exposed to doses slightly above natural background radiation levels. Predictions, generally based on the LNT [Linear No Threshold] model, suggest that up to 5000 additional cancer deaths may occur in this population from radiation exposure, or about 0.6% of the cancer deaths expected in this population due to other causes."

The above suggests that it is reasonable to assume that the distribution of ages is likely to be almost entirely independent of whether the community has or has not been exposed to low-level radiation. The steady state assumption then embodies the premise that the number of people in a community will stay approximately stable over time. The age distribution for such a community may be found as described in Appendix B.

In the case of the two most severe accidents experienced in the civil nuclear power sector, Chernobyl (1986) and Fukushima Daiichi (2011), large-scale relocation was applied to the public living close to the power plants, although the policy responses can be seen in hindsight to be excessive (Waddington et al., 2017). Additional studies using diverse methodologies (Yumashev et al., 2017; Ashley et al., 2017) suggest that the number of people relocated permanently after even a very severe accident at a modern nuclear power station (melt down and major release of radioactive material) ought, after considering the costs and the benefits, to be kept low in almost all cases; in many instances relocation emerges as an option to be avoided altogether.

The radiation dose administered as a result of a severe nuclear accident to a community staying in situ will, in fact, tend to fall over time as a result of radioactive decay and the continuing, gradual dispersal of radioactive fallout from man's environment (Waddington et al., 2017). The models used in this paper (point exposure and constant dose rate) may be regarded as limiting cases that bracket such a situation, an issue discussed at greater length and justified in Section 12.7 below.

Any continuing exposure of some years' duration may be presumed to have an effect on those not yet born when it started and this may be allowed for using the general approach outlined in Thomas et al. (2006). The starting age for the radiation cancer victims born while the radiation exposure is continuing will be 0, while the duration of the exposure they face will depend on their time of birth.

An individual born immediately after the start of the prolonged exposure (say the next day) will be subject to an exposure that lasts a time approaching $T_R$ years, while an individual born a year after the start of the prolonged exposure will see an exposure lasting $T_R - 1$ years. Generalising, the individual born $b$ years after the start of the prolonged exposure will see an exposure lasting $T_R - b$ years. Meanwhile the individual born $T_R$ years after the start of the prolonged exposure will see an exposure lasting $T_R - T_R = 0$ years, that is to say he will face no exposure.

We may write the effective period of exposure for someone born $b$ years after the start of a prolonged exposure as:

$$T_R^{(b)} = T_R - b \quad \text{for} \quad 0 \le b \le T_R \tag{62}$$

The number of individuals, $n_2$, who will be born into the exposure between $b$ and $b+1$ years into the exposure will be found by integrating the birth-rate, $n(0)$, over this one-year interval

$$n_2 = n(0) \int_{t=b}^{b+1} dt = n(0) = \frac{N}{X(0)} \tag{63}$$

where Eq. (B.4) from Appendix B has been used to give $n(0)$ in terms of the life expectancy at birth, $X(0)$, and the number in the population of size, $N$, is assumed to be in the steady state. A fraction of these will become radiation cancer victims with starting age, 0, where that fraction will be $c_v P(Y \le \alpha_0|0)$ by an analogy with Eq. (52) with the starting age set to zero: $a = 0$. Thus the density of radiation cancer victims born $b$ years after the start of the exposure will be:

$$n_{2v}^{(b)} = c_v n(0) P(Y \le \alpha_0|0) = \frac{c_v}{\Omega T_R^{(b)}} \frac{N}{X(0)} \psi_1^{\left(T_R^{(b)}\right)}(\alpha_0) f_A(0) = \frac{c_v}{\Omega(T_R - b)} \frac{N}{X(0)} \psi_1^{(T_R - b)}(\alpha_0) f_A(0) \tag{64}$$

where $T_R^{(b)}$ is the length of radiation exposure seen by someone born b years after the start of the exposure: $T_R^{(b)} = T_R - b$. Meanwhile $\psi_1^{\left(T_R^{(b)}\right)}(\alpha_0) = \int_{m=0}^{\alpha_0} \psi_0(m)\, dm$, in which $\psi_0(\tau)$ is given by Eq. (24), but now Eqs. (25) and (26) are replaced by $\omega_3 = T_R - b + \omega_1$ and $\omega_4 = T_R - b + \omega_2$.

The total number of victims born into the exposure will be:

$$N_{2\upsilon} = \int_{b=0}^{T_R} n_{2\upsilon}^{(b)} db = \frac{c_\upsilon N}{\Omega X(0)} f_A(0) \int_{b=0}^{T_R} \frac{\psi_1^{(T_R-b)}(\alpha_0)}{T_R - b} db \tag{65}$$

[Concerning the term in the last integrand, it may be noted that $\Psi_1^{(T_R-b)}/(T_R-b)$ does not tend to infinity as $b \to T_R$ but instead takes the constant value, $\Omega$. See Eq. (C.5) in Appendix C, which applies when $u = T_R - b \le \alpha_0 - \omega_2$, a condition satisfied when $b \to T_R$ so that $u \to 0$.]

The total number of radiation victims in the combined population of those living at the start of the radiation exposure and those born into the exposure is therefore

$$\begin{aligned} N_{T\upsilon} = N_\upsilon + N_{2\upsilon} &= \frac{c_\upsilon N}{\Omega T_R} \int_{a=0}^{\alpha_0-\omega_1} \psi_1(\alpha_0 - a) f_A(a)\, da + \frac{c_\upsilon N}{\Omega} \frac{1}{X(0)} f_A(0) \int_{b=0}^{T_R} \frac{\psi_1^{(T_R-b)}(\alpha_0)}{T_R-b} db \\ &= \frac{c_\upsilon N}{\Omega T_R} \left( \int_{a=0}^{\alpha_0-\omega_1} \psi_1(\alpha_0 - a) f_A(a)\, da + \frac{T_R}{X(0)} f_A(0) \int_{b=0}^{T_R} \frac{\psi_1^{(T_R-b)}(\alpha_0)}{T_R-b} db \right) \end{aligned} \tag{66}$$

where the expression for $N_\upsilon$ is that given in Eq. (53). The starting age distribution amongst this enlarged population will change as the period of radiation exposure increases. The probability density for starting age for the complete cohort of victims, $\pi_{\upsilon A}(a)$, is now given by:

$$\pi_{\upsilon A}(a) \approx \begin{cases} \dfrac{N_\upsilon(a)}{N_{T\upsilon}} & \text{for} \quad a > 0 \\[2ex] \dfrac{N_\upsilon(0) + N_{2\upsilon}}{N_{T\upsilon}} & \text{for} \quad a = 0 \end{cases} \tag{67}$$

The second line in Eq. (67) follows from the fact that, as noted above, all the people born during the time of the prolonged exposure will have a starting age of zero.

Hence substituting from Eqs. (52), (65) and (66) into Eq. (67) and cancelling the common term, $c_\upsilon N/\Omega T_R$, we achieve:

$$\pi_{\upsilon A}(a) \approx \begin{cases} \dfrac{\psi_1(\alpha_0 - a) f_A(a)}{\displaystyle\int_{a=0}^{\alpha_0-\omega_1} \psi_1(\alpha_0 - a) f_A(a)\, da + \frac{T_R}{X(0)} f_A(0) \int_{b=0}^{T_R} \frac{\psi_1^{(T_R-b)}(\alpha_0)}{T_R-b} db} & \text{for} \quad a \ge 0 \\[4ex] \dfrac{\psi_1(\alpha_0) f_A(0) + \displaystyle\frac{T_R}{X(0)} f_A(0) \int_{b=0}^{T_R} \frac{\psi_1^{(T_R-b)}(\alpha_0)}{T_R-b} db}{\displaystyle\int_{a=0}^{\alpha_0-\omega_1} \psi_1(\alpha_0 - a) f_A(a)\, da + \frac{T_R}{X(0)} f_A(0) \int_{b=0}^{T_R} \frac{\psi_1^{(T_R-b)}(\alpha_0)}{T_R-b} db} & \text{for} \quad a = 0 \end{cases} \tag{68}$$

The properties of the integral, $\int_{b=0}^{T_R} \frac{\psi_1^{(T_R-b)}(\alpha_0)}{T_R-b} db$, are discussed in Appendix C, which derives the following 3 formulae, depending on the length, $T_R$, of the radiation exposure:

$$\int_{b=0}^{T_R} \frac{\psi_1^{(T_R-b)}(\alpha_0)}{T_R-b} db = \Omega T_R \qquad T_R \le \alpha_0 - \omega_2 \tag{69}$$

$$\int_{b=0}^{T_R} \frac{\psi_1^{(T_R-b)}(\alpha_0)}{T_R-b} db = \Omega(\alpha_0 - \omega_2) + \frac{1}{4}(T_R - (\alpha_0 - \omega_2))(3(\alpha_0 - \omega_1) + \Omega - T_R) - \frac{(\alpha_0 - \omega_2)^2}{2} \ln\left(\frac{T_R}{\alpha_0 - \omega_2}\right) \quad \text{for} \quad \alpha_0 - \omega_2 < T_R \le \alpha_0 - \omega_1 \tag{70}$$

$$\int_{b=0}^{T_R} \frac{\psi_1^{(T_R-b)}(\alpha_0)}{T_R-b} db = \Omega(\alpha_0 - \omega_2) + \frac{\Omega}{4}(2\alpha_0 - 3\omega_1 + \omega_2) - \frac{(\alpha_0 - \omega_2)^2}{2} \ln\frac{\alpha_0 - \omega_1}{\alpha_0 - \omega_2} + \Omega\left(\alpha_0 - \frac{\omega_1 + \omega_2}{2}\right) \ln\left(\frac{T_R}{\alpha_0 - \omega_1}\right) \qquad T_R > \alpha_0 - \omega_1 \tag{71}$$

## 9. Average age at death for radiation-cancer victims, both those living at the start of the exposure and those born into it

It is now possible to extend the treatment and deal not only with those living at the start of the continuing dose, but also those born during the period when the exposure was going on. This will extend the results established in Section 7, which applied only to the people living at the start of exposure.

The mean age at death for a radiation cancer victim born $b$ years after the start of the radiation exposure is given by using Eq. (56) and setting starting age, $a$, to zero in Eq. (41) to give:

$$E(Y) = E(Y|A=0) = \frac{\theta_1(\alpha_0)}{\psi_1(\alpha_0)} \quad \text{for} \quad 0 \le b < T_R \tag{72}$$

Similarly, putting $a = 0$ in Eq. (46) gives the variance of the age at death for a radiation cancer victim born $b$ years after the start of the radiation exposure:

$$var(Y) = var(Y|A=0) = \frac{\theta_2(\alpha_0)}{\psi_1(\alpha_0)} - \left(\frac{\theta_1(\alpha_0)}{\psi_1(\alpha_0)}\right)^2 \quad \text{for} \quad 0 \le b < T_R \tag{73}$$

To find the mean and variance of the age at death for the inclusive cohort consisting of both those living at the start of the exposure and those born into it, the equations developed in Section 4 may now be applied, except that the probability density, $f_{vA}(a)$, will be replaced by $\pi_{vA}(a)$. Thus the overall expected age at death for radiation cancer victims when the exposure lasts $T_R$ years comes from replacing Eq. (56) with

$$E(Y) = E(E(Y|A)) = \int_{a=0}^{\alpha_0-\omega_1} \pi_{vA}(a) E(Y|A=a)\, da \tag{74}$$

or, using Eq. (41):

$$E(Y) = \int_{a=0}^{\alpha_0-\omega_1} \pi_{vA}(a)\left(a + \frac{\theta_1(\alpha_0-a)}{\psi_1(\alpha_0-a)}\right) da = E(A) + E\left(\frac{\theta_1(\alpha_0-A)}{\psi_1(\alpha_0-A)}\right) \tag{75}$$

where $E(A)$ is the average starting age of the combined population of radiation cancers victims, both those living at the start of the radiation exposure and those born during the exposure:

$$E(A) = \int_{a=0}^{\alpha_0-\omega_1} \pi_{vA}(a)\, a da \tag{76}$$

while the term, $E(\theta_1(\alpha_0-A)/\psi_1(\alpha_0-A))$, represents the additional amount needed to give the average age at death for this extended cohort:

$$E\left(\frac{\theta_1(\alpha_0-A)}{\psi_1(\alpha_0-A)}\right) = \int_{a=0}^{\alpha_0-\omega_1} \pi_{vA}(a) \frac{\theta_1(\alpha_0-a)}{\psi_1(\alpha_0-a)} da \tag{77}$$

Similarly Eq. (59) becomes:

$$var(E(Y|A)) = \int_{a=0}^{\alpha_0-\omega_1} \pi_{vA}(a)(E(Y|A=a))^2 da - [E(Y)]^2 = \int_{a=0}^{\alpha_0-\omega_1} \pi_{vA}(a)\left(a + \frac{\theta_1(\alpha_0-a)}{\psi_1(\alpha_0-a)}\right)^2 da - [E(Y)]^2 \tag{78}$$

where Eq. (41) has been used in the second step, and $[E(Y)]$ is given by Eq. (72).

Finally, Eq. (60) becomes:

$$E(var(Y|A)) = \int_{a=0}^{\alpha_0-\omega_1} \pi_{vA}(a)\left(\frac{\theta_2(\alpha_0-a)}{\psi_1(\alpha_0-a)} - \left(\frac{\theta_1(\alpha_0-a)}{\psi_1(\alpha_0-a)}\right)^2\right) da \tag{79}$$

The variance associated with the age at death for radiation cancer victims exposed for $T_R$ years, $var(Y)$, may then be calculated by substituting from Eqs. (78) and (79) into Eq. (61).

## 10. Point radiation dose

As noted above, the case of a one-off radiation dose occurring at a single point in time may be approximated assuming the radiation dose occurs over a small period, say 0.01 years. However, it is also possible to consider the basic equation governing the age at death for a radiation victim who experiences a one-off dose, namely:

$$Y = A + M \tag{80}$$

where $A$ is the age of the person, chosen randomly from the population, $M$ is the mortality period, a random variable, and $Y$ is the age at death. This is the form that Eq. (15) will take when the age at cancer induction, $X$, is equal to the starting age, $A$: $X = A$, as must be the case for a radiation cancer victim exposed to a point dose only. In the more general case considered earlier of a continuing rather than point dose, the corresponding relationship is $X \geq A$.

From Eq. (18), the expected value, $E(M)$, of mortality period at low starting ages is simply $(\omega_1 + \omega_2)/2$ years. However, by the model used, there will be an earlier cut-off of the period of risk for those aged more than 61, since the effective upper point of the uniform distribution will not be $\omega_2$ but $\alpha_0 - a$ instead, where $a$ is the starting age, in this case the single age at which exposure occurs. Thus the mortality period used in the Marshall model is subject to limits that vary with starting age:

$$\begin{aligned} &\omega_1 \leq M < \omega_2 && \text{for} \quad 0 \leq a < \alpha_0 - \omega_2 \\ &\omega_1 \leq M < \alpha_0 - \omega_1 - a && \text{for} \quad \alpha_0 - \omega_2 \leq a < \alpha_0 - \omega_1 \end{aligned} \tag{81}$$

The distribution is uniform between the upper and lower limits in both instances, so that we may partition the probability density for mortality period as:

$$f_M(m) = \begin{cases} \dfrac{1}{\omega_2 - \omega_1} & \text{for } 0 \leq a < \alpha_0 - \omega_2 \\ \dfrac{1}{\alpha_0 - a - \omega_1} & \text{for} \quad \alpha_0 - \omega_2 \leq a < \alpha_0 - \omega_1 \end{cases} \tag{82}$$

Since the distribution is uniform between limits, the expected value, $E(M|A = a)$ is given by

$$E(M|A = a) = \begin{cases} \dfrac{\omega_1 + \omega_2}{2} & \text{for} \quad 0 \leq a < \alpha_0 - \omega_2 \\ \dfrac{\alpha_0 + a + \omega_1}{2} & \text{for} \quad \alpha_0 - \omega_2 \leq a \leq \alpha_0 - \omega_1 \end{cases} \tag{83}$$

The expected value of age at death for a radiation cancer victim of starting age, $a$, is:

$$E(Y|A = a) = E(M|A = a) + E(A|A = a) = E(M|A = a) + a \tag{84}$$

The expected value of age at death for all radiation cancer victims may then be computed from the integral

$$E(Y) = E(E(Y|A)) = \int_{a=0}^{\alpha_0-\omega_1} f_{vA}(a)\, E(Y|A = a)\, da = \int_{a=0}^{\alpha_0-\omega_1} a f_{vA}(a)\, da + \int_{a=0}^{\alpha_0-\omega_1} f_{vA}(a)\, E(M|A = a)\, da = E(A) + E(E(M|A = a)) \tag{85}$$

where the expected values are evaluated over the range of starting ages appropriate for radiation cancer victims, namely: $0 \leq a \leq \alpha_0 - \omega_1$.

The variance of the mortality period for a starting age, $a$, is found by integrating between the appropriate mortality period limits, which will depend on the value of the starting age, $a$, as given in Eq. (81). For $0 \leq a < \alpha_0 - \omega_2$:

$$\begin{aligned} var(M|A = a) &= \frac{1}{\omega_2 - \omega_1} \int_{m=\omega_1}^{\omega_2} m^2 dm - (E(M|A = a))^2 \\ &= \frac{\omega_2^3 - \omega_1^3}{3(\omega_2 - \omega_1)} - \left(\frac{\omega_2 + \omega_1}{2}\right)^2 \qquad \text{for } 0 \leq a < \alpha_0 - \omega_2 \\ &= \frac{1}{12}(\omega_2 - \omega_1)^2 \end{aligned} \tag{86}$$

Meanwhile for starting ages in the range: $\alpha_0 - \omega_2 \le a < \alpha_0 - \omega_1$,

$$\begin{aligned} var\,(M|A = a) &= \frac{1}{\alpha_0 - a - \omega_1} \int_{m=\omega_1}^{\alpha_0-\omega_1-a} m^2 dm - (E\,(M|A = a))^2 \\ &= \frac{(\alpha_0 - \omega_1 - a)^3 - \omega_1^3}{3\alpha_0 - a - \omega_1} - \left(\frac{\alpha_0 + a + \omega_1}{2}\right)^2 \qquad \alpha_0 - \omega_2 \le a < \alpha_0 - \omega_1 \\ &= \frac{1}{12}((\alpha_0 - a - \omega_1) - \omega_1)^2 \end{aligned} \tag{87}$$

But, for any given starting age, $A = a$, the age at death for a radiation cancer victim will be $Y = a + M$, so that the variances shown above will also characterise the age at death, $Y$, given $A = a$. Hence we may write:

$$var\,(Y|A = a) = \begin{cases} \frac{1}{12}(\omega_2 - \omega_1)^2 & 0 \le a < \alpha_0 - \omega_2 \\ \frac{1}{12}(\alpha_0 - a - 2\omega_1)^2 & \alpha_0 - \omega_2 \le a < \alpha_0 - \omega_1 \end{cases} \tag{88}$$

The term, $E\,(var\,(Y|A))$, may be found by taking the expectation value over the range of starting ages appropriate for radiation cancer victims:

$$E\,(var\,(Y|A = a)) = \int_{a=0}^{\alpha_0-\omega_1} var\,(Y|A = a)\, f_{vA}\,(a)\, da = \frac{1}{12}(\omega_2 - \omega_1)^2 \int_{a=0}^{\alpha_0-\omega_2} f_{vA}\,(a)\, da + \frac{1}{12} \int_{a=\alpha_0-\omega_2}^{\alpha_0-\omega_1} (\alpha_0 - a - 2\omega_1)^2 f_{vA}\,(a)\, da \tag{89}$$

The term, $var\,(E\,(Y|A))$ may be then computed by applying Eq. (59) to Eqs. (83) and (84), using $E\,(Y)$ from Eq. (85). The variance for age at death, $var\,(Y)$, may then be calculated using Eq. (61), allowing the standard deviation to be found by taking the square root.

## 11. Unconditional probability distribution for age at death for radiation cancer victims

The methods described allow the mean and variance for age at death to be found both for a radiation cancer victim of a specified starting age and also for the whole cohort of radiation cancer victims in the general population, including those not born when the exposure started. The fact that the mean and the variance are known allows limits to be placed on the likely range of ages at death using Chebyshev's inequality, which states that the probability of a random variable, $V$, lying more than $n$ standard deviations, $\sigma$, away from its mean value, $\mu$, is no more than $1/n^2$. The complementary probability that the variable lies within the limits $\mu \pm n\sigma$ must be at least $1 - 1/n^2$:

$$P\,(|V - \mu| < n\sigma) \ge 1 - \frac{1}{n^2} \tag{90}$$

Putting $\sigma_Y = \sqrt{var\,(Y)}$ and using Eq. (90), the probability that the age at death, $Y$, lies within the range, $E\,(Y) \pm 1.58\sigma_Y$, for example, is at least 60%, irrespective of whatever probability distribution may characterise $Y$. The generality of Chebyshev's inequality means that it could be applied to all models discussed in the previous sections.

Nevertheless it is desirable to give more exact limits, something that is enabled by the following treatment, which builds on previous results. The probability of a radiation cancer victim having a starting age in the vicinity of $a$ and then dying in the vicinity of age $y$ will be $\pi_{vA}\,(a)\, da \times f_{Y|A}\,(y|a)\, dy$, where $\pi_{vA}\,(a)$ is given by Eq. (68) and $f_{Y|A}\,(y|a)$ is the probability density given by Eq. (38). The total probability for age at death in the vicinity of $y$ is thus $f_Y\,(y)\, dy$, found by integrating the individual components:

$$f_Y\,(y)\, dy = \int_{\text{all starting ages, } a} \pi_{vA}\,(a)\, f_{Y|A}\,(y|a)\, da \times dy \tag{91}$$

Hence

$$f_Y\,(y) = \int_{a=0}^{\alpha_0-\omega_1} \pi_{vA}\,(a)\, f_{Y|A}\,(y|a)\, da \tag{92}$$

The cumulative probability up to a selected value, $y_s$, may then be found from:

$$P\,(Y < y_s) = \int_{y=0}^{y_s} f_Y\,(y)\, dy \tag{93}$$

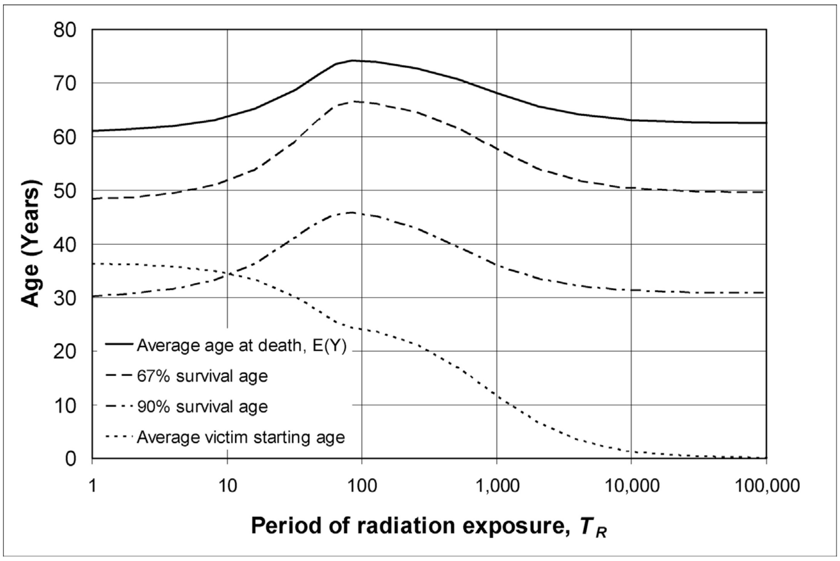


**Fig. 5 – Mean age at death, $E(Y)$ vs. period, $T_R$, of exposure. 67% and 90% survival ages and the average starting age, $E(A)$, for radiation cancer victims.**

Once $f_Y(y)$ has been established, the expected value of age at death for a radiation cancer victim may be evaluated as:

$$E(Y) = \int_{y=0}^{\alpha_0} y f_Y(y)\, dy \tag{94}$$

and similarly the variance:

$$var(Y) = \int_{y=0}^{\alpha_0} y^2 f_Y(y)\, dy - (E(Y))^2 \tag{95}$$

These equations offer a diverse check on the previous results for a prolonged radiation exposure that culminated in Eq. (56) and the grouping of Eqs. (59)–(61). Moreover, setting the period, $T_R$, of exposure to a low value allows the results for a point exposure, to be checked (Eqs. (84), (85), (59), (89) and (61)).

## 12. Results

### 12.1. *Radiation cancer victims of all starting ages*

The results apply to notional radiation cancer victims within the general public in the UK, being based on the UK life tables, 2012–2014 (Office for National Statistics, 2015).

The average age at death, $E(Y)$, for the cohort of all radiation cancer victims will vary with the length of the period of exposure, $T_R$, as will the variance, $var(Y)$. Fig. 5 shows the mean age at death, which begins at 61.1 years when $T_R = 1$, reaches a maximum of 74.2 years when $T_R = 85$ years and then falls back to 62.5 years as the exposure period tends to infinity. The reason for this peak in average age at death is explored at length in this section, starting from the next paragraph. The figure also shows the 67% survival age, where the chance of living longer is more than double that of dying younger. This starts at 48.3 years, rises to 66.6 years when $T_R = 85$ and then declines to 49.6 years for very long exposures. Fig. 5 displays, in addition, the 90% survival age, when the probability of living longer is nine times that of dying younger. This begins at 30.3 years, peaks at 45.8 years when $T_R = 85$ and then drops back to 30.9 years.

It is possible by examining the unconditional probability density for age at death, $f_Y(y)$, to explain the gradual increase, peak and then partial decline in the average age at death, $y$, as the period, $T_R$, of radiation exposure increases. Fig. 6 shows that when $T_R = 1$, the curve $f_Y(y)$ encloses roughly equal areas above and below the mean age of death, 61 years. The median is just less than the mean for this short period of exposure. The general shape of the probability density for age at death remains reasonably similar when $T_R$ has increased to 30 years (see Fig. 7), but the graph has shifted towards higher ages at death and the median now lies above the mean. Fig. 8 shows the continuation of this upward trend when $T_R$ has increased to 60 years. The average age at death reaches its maximum (74.2 years) when the period of exposure is 85 years, when the probability density resembles a right-angled triangle (see Fig. 9). The reason why this probability density takes on this shape may be understood by considering the conditional probability densities, $f_{Y|A}(y|a)$, a weighted integral of which constitutes the unconditional probability density for age at death, $f_Y(y)$ (see Eq. (92)).

Fig. 10 shows the conditional probability densities, $f_{Y|A}(y|a)$, for starting ages, $a$, of: 0, 20, 40 and 60 years. Since $f_{Y|A}(y|a) \propto \psi_0(y-a)$ by Eq. (38), the conditional probability density curve will be geometrically similar to the shape of $\psi_0(y-a)$, which is

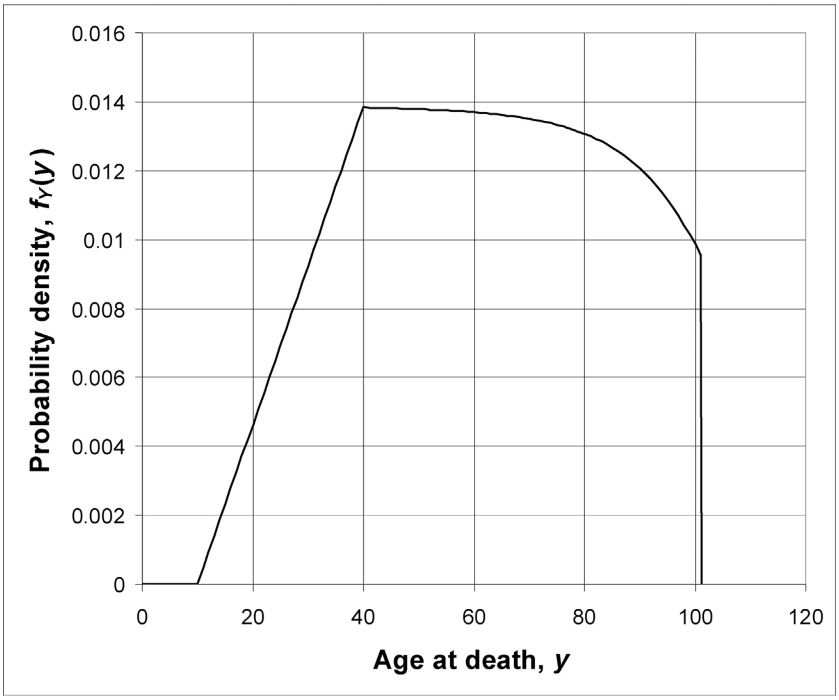


**Fig. 6 – Probability density, $f_Y(y)$, for age at death for radiation cancer victims when the exposure period is one year: $T_R = 1$.**

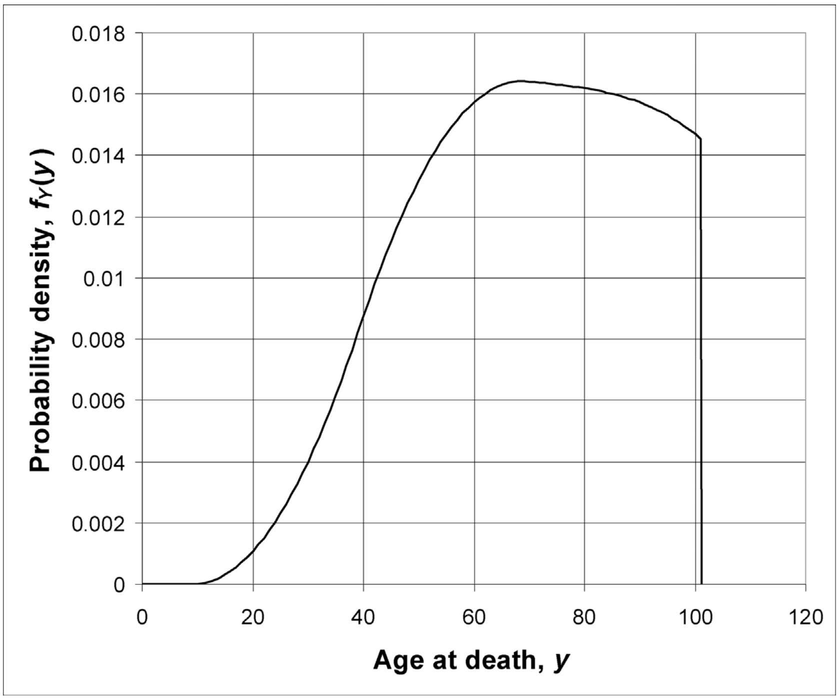


**Fig. 7 – Probability density, $f_Y(y)$, for age at death for radiation cancer victims when the exposure period is 30 years: $T_R = 30$.**

characterised in Fig. 4 as one of a set of truncated, flat-topped isosceles triangles. Provided the maximum age at death, y, lies below the upper age limit, $\alpha_0$, then $\psi_0(y - a)$ and hence $f_{Y|A}(y|a)$ will display at least part of the descending leg observable at the upper end of the truncated triangles. This requirement, max $y < \alpha_0$, implies that $a + \omega_3 < \alpha_0$, or, since $\omega_3 = \omega_1 + T_R$

$$T_R < \alpha_0 - \omega_1 - a \quad (96)$$

The condition ceases to be met for $T_R = 85$ when the starting age reaches 6 years (as $\alpha_0 - \omega_1 - a = 101 - 10 - 6 = 85$), and this causes the curve for $f_{Y|A}(y|a)$ for $a \geq 6$ to be devoid of a descending leg but to terminate instead on the upper plateau. Such behaviour is shown for 3 out of the 4 curves in Fig. 10, with just a small portion of the downward leg still visible for the fourth curve, $f_{Y|A}(y|a)$, valid when the starting age is zero: $a = 0$.

The curves, $f_{Y|A}(y|a)$, do not change past $T_R > 91$ since, for all starting ages, $a$, they will have passed what may be called the "saturation point" for the exposure period: $T_R = \alpha_0 - \omega_1 - a$. Thus none will possess even the vestiges of a descending leg and all will terminate on the upper plateau.

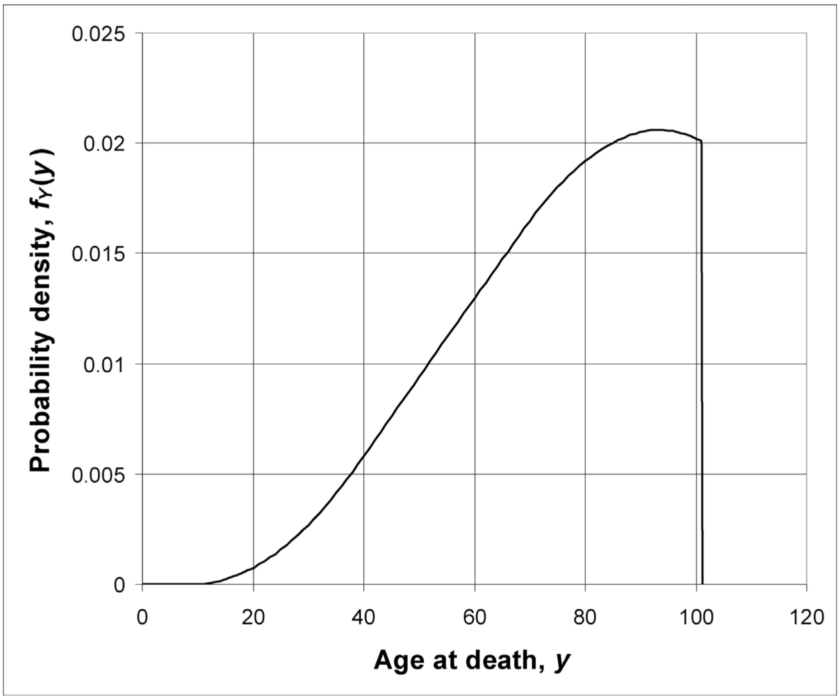


**Fig. 8 – Probability density, $f_Y$ (y), for age at death for radiation cancer victims when the exposure period is 60 years: $T_R = 60$.**

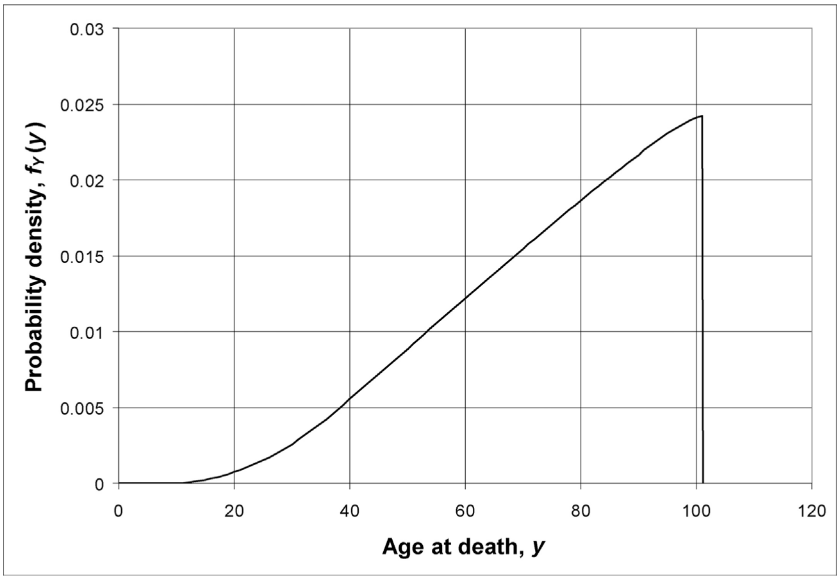


**Fig. 9 – Probability density, $f_Y$ (y), for age at death for radiation cancer victims when the exposure period is 85 years: $T_R = 85$.**

Taking the curve, $f_{Y|A}(y|0)$, formed when the starting age is zero, the beginnings of the downward leg seen in Fig. 10 when $T_R = 85$, will be replaced, when $T_R > 91$, by an extension of the upper plateau between $y = 95$ and $y = 101$, thus giving a greater emphasis to these higher ages at death. See Fig. 11, which shows the conditional probability densities, $f_{Y|A}(y|a)$, when $T_R = 100$ years.

As a thought experiment, consider what would happen if the weightings, $\pi_{\upsilon A}(a)$, applied to the conditional probability densities, $f_{Y|A}(y|a)$, were independent of period of exposure, $T_R$, for example if $\pi_{\upsilon A}(a) = f_{\upsilon A}(a)$ in Eq. (92). In such a case the weighting given to the higher ages of death, $y$, in calculating $E(Y)$ would be reach a maximum at $T_R = 91$. Hence the mean age at death, $E(Y)$, would plateau out at its highest value when the exposure period reached 91 years. This maximum would then hold for all $T_R$ greater than 91 years.

In fact, as Eq. (68) tells us, the probability density, $\pi_{\upsilon A}(a)$, will change continuously with $T_R$, giving ever greater weight to the lowest possible value of starting age, $a = 0$, as the period of exposure increases. This acts as a countervailing influence on $E(Y)$, which peaks when the period of exposure is $T_R = 85$ years, just before the 91 years discussed in the previous paragraph. The expected age at death for a radiation cancer victim, $E(Y)$, then falls. It will converge on the expected value for age at death for a person with a starting age of zero when the exposure period is 91 years or more, that is to say

$$E(Y) \rightarrow E(Y|0)|_{T_R \geq 91} \quad \text{as} \quad T_R \rightarrow \infty \tag{97}$$

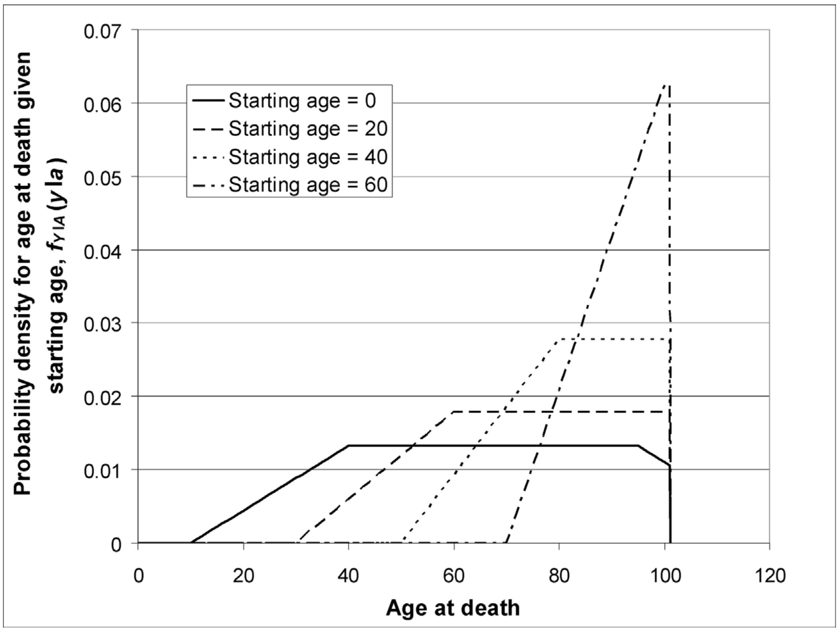


**Fig. 10 – Conditional probability densities, $f_{Y|A}(y|a)$, for starting ages, $a$, of 0, 20, 40 and 60 years. Period of exposure, $T_R$, is 85 years.**

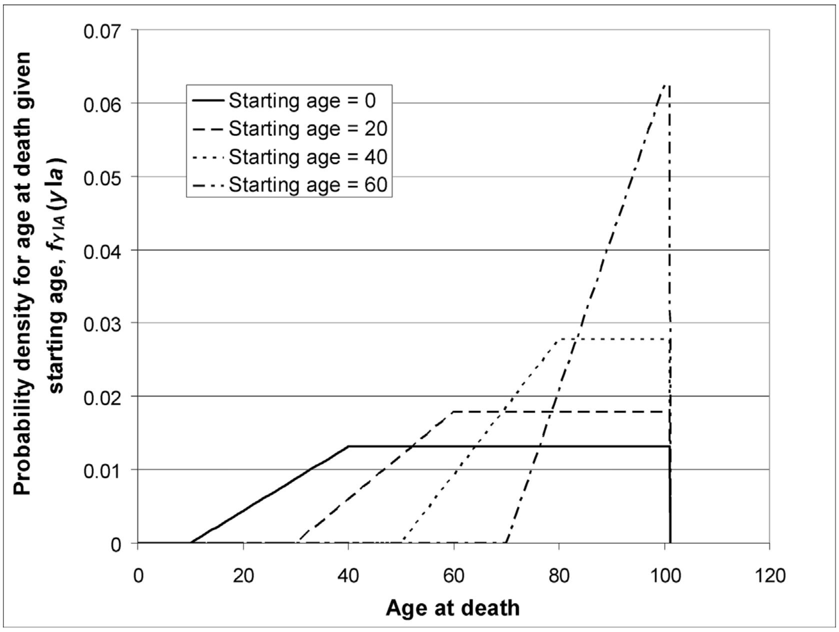


**Fig. 11 – Conditional probability densities, $f_{Y|A}(y|a)$, for starting ages, $a$, of 0, 20, 40 and 60 years. Period of exposure, $T_R$, is 100 years.**

The unconditional probability density, $f_Y(y)$, will then be the same as the conditional probability density, $f_{Y|A}(y|0)$, for a person with starting age, $a = 0$. This may be seen by comparing the full line in the graph of Fig. 11, depicting the behaviour of $f_{Y|A}(y|0)$, with the unconditional probability density, $f_Y(y)$, shown in Fig. 12 for $T_R$ set at a million years as a proxy for infinity.

Fig. 5 also displays the mean starting age, $E(A)$, for radiation cancer victims, which is 36.5 years when $T_R = 1$. This figure lies about 5 years below the average age (41.7 years) of the general public because those exposed to the same dose at higher ages will have a smaller opportunity to develop a radiation induced cancer, no chance at all if their starting age is above 91. The mean starting age then falls when $T_R$ increases, as an ever growing fraction of the total cohort of victims are born into the exposure. As shown in Fig. 13, the probability density for victims with a zero starting age, $\pi_{vA}(0)$, approaches unity when the exposure is constant over all future time: $\pi_{vA}(0) \to 1$ as $T_R \to \infty$, as intuition would suggest.

### *12.2. One-off dose*

Fig. 14 is produced when the methods of Section 10 are applied and the resultant values for mean (60.6 years) and standard deviation (22.8 years) are plotted on the same graph as the corresponding values for a prolonged exposure. Clearly an ideal point exposure has $T_R \to 0$, but a duration of $T_R = 0.01$ years was chosen to conform to the requirements of a logarithmic scale. The mean value is within 0.3% of that found from that found by the prolonged-exposure method with $T_R = 0.01$ years, while the

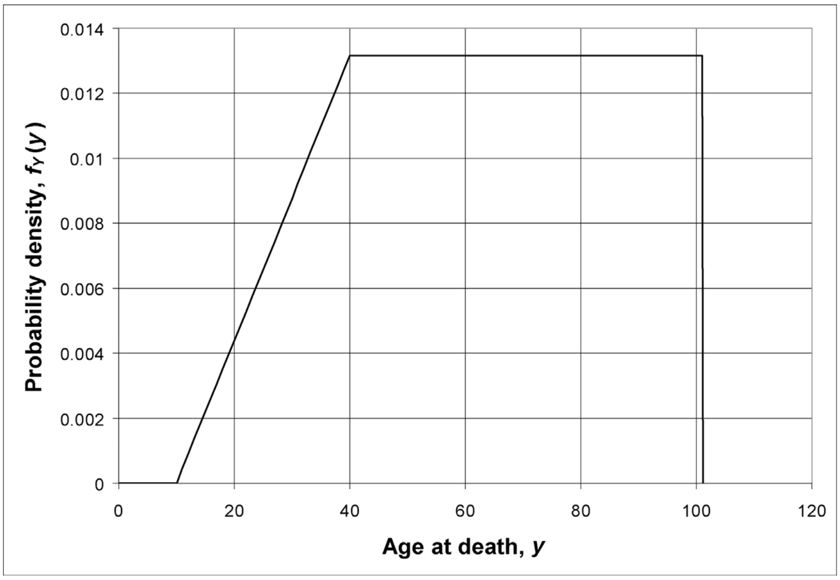


**Fig. 12 – Probability density, $f_Y(y)$. Period of exposure, $T_R$, is one million years.**

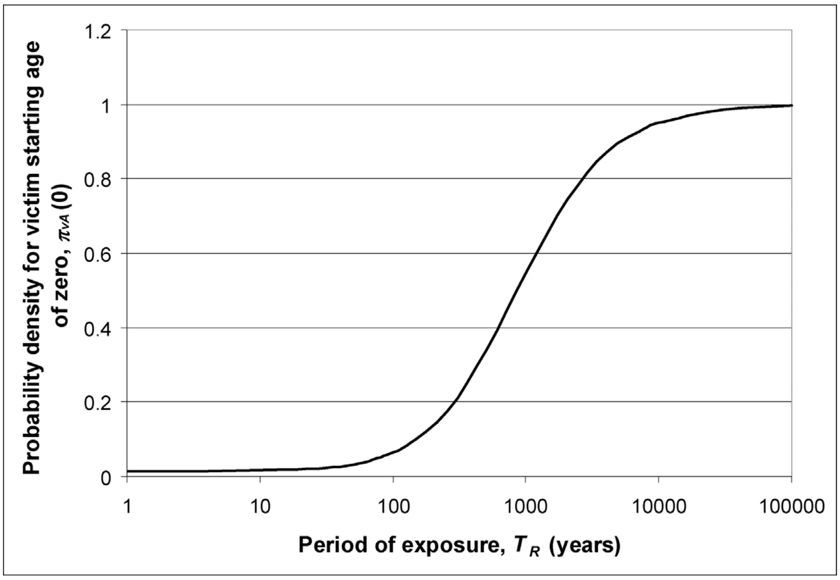


**Fig. 13 – Probability density for a zero starting age amongst the total cohort of radiation cancer victims.**

standard deviation is within 1.5%. This is regarded as a reasonable match given that each result depends on several numerical integrations where the step length is constrained by the data to be 1 year. Thus a diverse check is provided on the correctness of the analysis. It is clear that the general trend apparent with prolonged exposures is maintained for both the mean and the standard deviation resulting from the one-off exposure.

### 12.3. *Radiation cancer victims with a starting age of zero*

This cohort consists of radiation cancer victims newly born when the exposure begins and those born into the exposure. Of all the age groups of radiation cancer victims, this group with a starting age, $a = 0$, will face the lowest life expectancy. Their average length of life is shown against the duration of the exposure in Fig. 15. This starts at 25.5 years for an exposure lasting 1 year and then climbs linearly with the radiation period, $T_R$, until $T_R = 61$ years, after which the rise in age at death slows before a plateau is reached when $T_R = 91$ years, after which the average age of death for a radiation cancer victim born into the exposure stays constant at 62.5 years. Although a small number of these victims will die as a result of the radiation dose they received in their first year of life, most victims will contract their radiation-induced cancer after that time. Given that they are victims, they are bound to die from a radiation cancer induced at some point, but when the exposure is very long, radiation victims with a starting age of zero may contract the disease over the full range of possible ages, 0 to 91 years.

Fig. 15 shows also the 67% and 90% survival ages. 67% of the cohort will live to be 19.9 years or more when the exposure period is 1 year, a survival age that rises 49.6 years when the exposure lasts 91 years or more. Meanwhile 90% of the cohort will die after the age of 13 when the exposure period is 1 year, a figure that rises to 31 years for an exposure lasting 91 years or more.


**Fig. 14 – Mean age at death and standard deviation for age at death: the figures for a one-off dose have been added in at $T_R = 0.01$ years.**

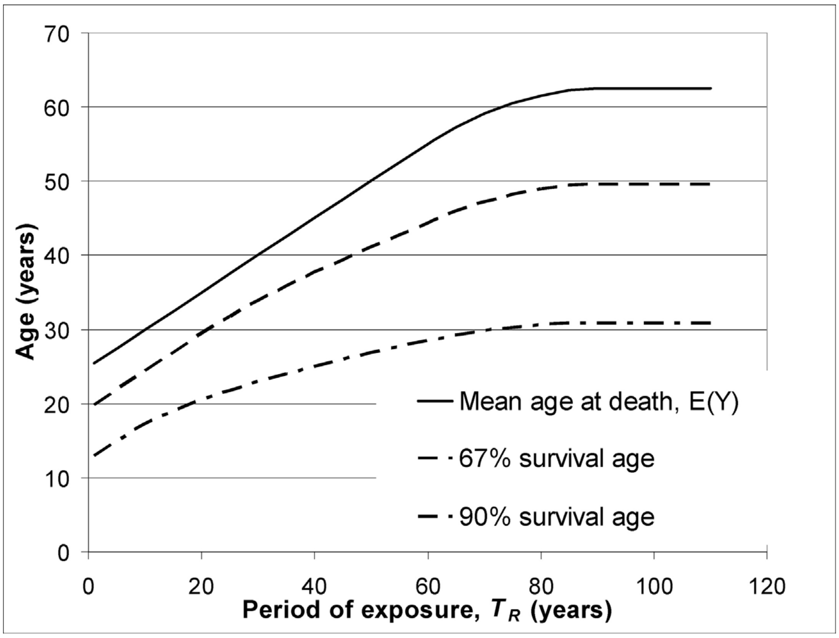


**Fig. 15 – Age at death for radiation cancer victims with a starting age of zero.**

### 12.4. *Life expectancy of the radiation cancer victims*

An individual's age at death, $Y$, is related to his or her starting age, $A$, and life to come from that age, $L$, by

$$Y = A + L \tag{98}$$

Applying the expectation operator produces

$$E(Y) = E(A) + E(L) \tag{99}$$

Hence the life expectancy for the cohort of radiation cancer victims is simply $E(L) = E(Y) - E(A)$, where both figures are evaluated using the analysis above.

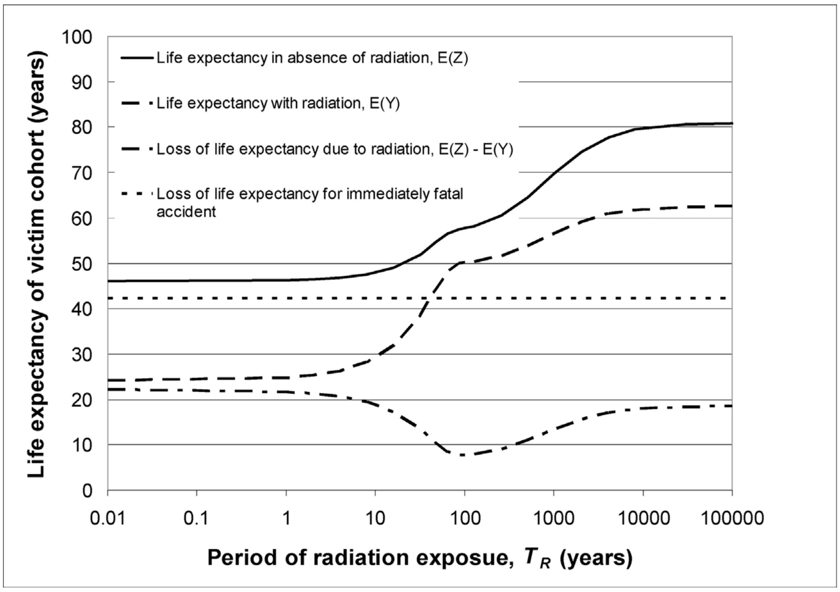


**Fig. 16 – Life expectancy for cohort of radiation cancer victims exposed to radiation for a period, $T_R$ years and what the life expectancy for this cohort would have been in the absence of radiation. Also the resultant loss of life expectancy due to radiation compared with the life expectancy from an immediately fatal accident.**

It is also possible to find the average expectation of life, $E(Z)$, that these victims would have had if they had not been exposed to a radiation cancer as:

$$E(Z) = \begin{cases} \displaystyle\int_{a=0}^{\alpha_0-\omega_1} x(a) f_{\upsilon A}(a)\, da & \text{for a one-off exposure} \\ \displaystyle\int_{a=0}^{\alpha_0-\omega_1} x(a) \pi_{\upsilon A}(a)\, da & \text{for a continuing exposure} \end{cases} \tag{100}$$

where $x(a)$ is the life expectancy at age, $a$, as given by the life tables. Were the radiation exposure to be entirely eliminated, the life expectancy, $E(L^*)$, of these notional radiation cancer victims would be $E(L^*) = E(Z) - E(A)$

The average age of the victims will be below that of the population at the time of first exposure, and this means that the average life expectancy of those who would become radiation cancer victims will, in the absence of radiation, always be greater that of the general population at the time of first exposure. For example, the life expectancy of the UK population may be calculated as 42.2 years, while if a point radiation exposure were to be eliminated, the potential victims of such a point exposure would have a life expectancy 46.1 years. The age of the cohort of notional victims will decrease as the length of the exposure increases, as discussed previously, and this will lead to $E(Z)$ increasing towards the life expectancy at birth, just under 81 years in the UK. This behaviour is shown in Fig. 16, which uses the convention discussed in Section 12.2, whereby the one-off figures are plotted against a radiation exposure period of 0.01 years. The life expectancy lost as a result of the radiation exposure is the difference in life expectancies:

$$E(L^*) - E(L) = E(Z) - E(A) - (E(Y) - E(A)) = E(Z) - E(Y) \tag{101}$$

This is also shown on the graph of Fig. 16, which shows that this radiation-generated loss of life expectancy for the cohort of victims starts at 22 years for a point exposure, reduces to 7.6 years when the radiation exposure lasts 85 years and then climbs back to 18.5 years for an ever-present radiation exposure.

Fig. 16 also shows for comparative purposes the loss of life expectancy a person chosen at random from the general population would expect to lose if he or she fell victim to an immediately fatal accident such as a fatal car crash or a train accident that killed a number of passengers. It is legitimate to compare this figure, $X = 42.2$ years, with the loss of life expectancy for the victims of a prolonged radiation exposure for the following reason. Suppose that the number of people falling victim to death from radiation cancer due to an exposure of length $T_R$ years is $m$. Their expected total loss of life expectancy will be $m \times E(L^*) = m(E(Z) - E(Y))$. If the same number of people died in an immediately fatal accident that could affect any member of the population, the expected total loss of life expectancy of these people would be $mX$, where $X$ is the population life expectancy. Thus the loss of life expectancy per victim would be $X$ in the case of an immediately fatal accident and $E(L^*)$ in the case of a radiation exposure of length, $T_R$. Taking the ratio, $E(L^*)/X$, it is clear that the radiation victim's loss of life expectancy is 52% of the loss from an immediately fatal accident if the radiation dose comes in a point exposure. This figure decreases to 18% as the radiation exposure increases in length to 85 years, and then steadies out at 44% for an extremely long exposure.

It will not be possible to identify the potential radiation cancer victims in advance. However, for completeness, we may consider the thought experiment whereby all the potential radiation victims can be identified and the choice is given to them, or

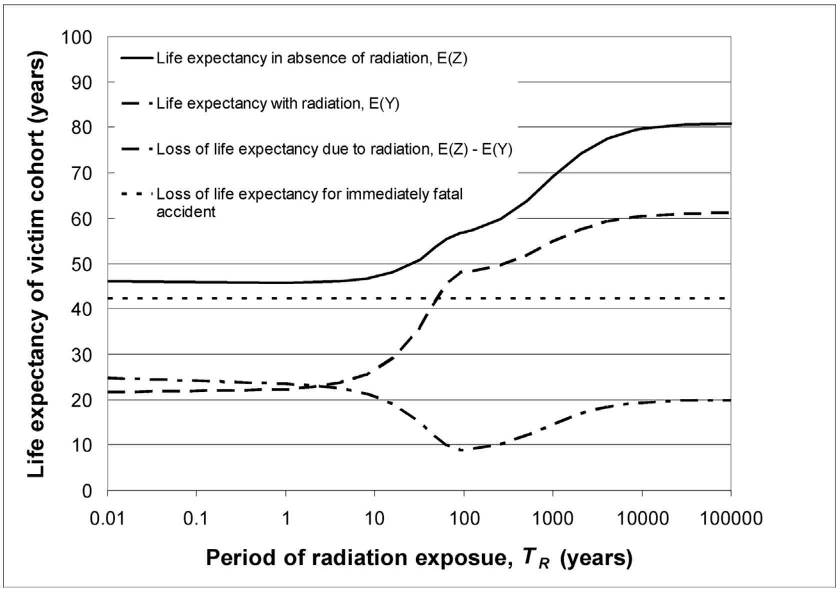


**Fig. 17 – Sensitivity Study 1: $\omega$1 reduced from 10 years to 5 years.**
**Life expectancy for cohort of radiation cancer victims exposed to radiation for a period, $T_R$ years; life expectancy for this cohort in the absence of radiation. Resultant loss of life expectancy due to radiation compared with the life expectancy from an immediately fatal accident.**

to adult representatives in the current generation acting for them, of whether they would prefer to die as a result of a radiation dose or in an immediately fatal accident that occurred at the start of the radiation dose. We know from Fig. 5 that the starting age of the cohort of radiation victims will always be less than the average age, 41.7 years for the UK, of the population, which will mean that their life expectancy in the absence of radiation will be longer, as shown by the unbroken line in Fig. 16. It is this life expectancy, $E(Z)$, which starts at 46.1 years and climbs to the life expectancy, 81 years, characteristic of the newly born, that would be taken away if the victims' radiation exposure were to be replaced by an accident with immediately fatal effects. Forming the ratio, $E(L^*)/E(Z)$, we see that the radiation victim's loss of life expectancy starts at 48% of the loss from an immediately fatal accident for a point exposure, decreases to 13% at a radiation exposure period of 85 years, and then steadies out at 23% for an extremely long exposure. Clearly the life that a radiation cancer victim can expect to benefit from is much longer than would be the case if he or she were subjected to an immediately fatal accident. This provides a strong reason for not selecting an immediately fatal accident as the preferred option, were it possible to offer potential radiation cancer victims this invidious choice.

### 12.5. *Sensitivity Study 1: the delay before a radiation cancer death can be caused is reduced by 5 years: $\omega_1 = 5$*

The results in Sensitivity Study 1 are generally similar to those in the main study. The mean age at death for a radiation cancer victim is down 1.9 years at 59.2 years when $T_R = 1$ year; it reaches a maximum of 73.1 years, down 1.1 years, at $T_R = 90$ years, and then comes back to 61.1 years for an extremely long exposure, down 1.4 years on the base case.

Fig. 17 may be compared directly with Fig. 16. It shows the life expectancy in the cohort of victims given a radiation exposure of $T_R$ years, their life expectancy if that radiation were to be completely averted and the difference between the two, which represents the loss of life expectancy for a given value of $T_R$. The loss of life expectancy caused by radiation starts at 24.6 years for a point exposure, up 2.6 years from the base case, reaches a minimum at $T_R = 90$ of 8.8 years, up 1.2 years, and then steadies out for an extremely long exposure at 19.9 years, an increase of 1.4 years. The corresponding percentages of the loss of life expectancy caused by a fatal accident occurring at the same time as the radiation cancer victim is first exposed to radiation are 58.3%, 20.9% and 47.1%.

### 12.6. *Sensitivity Study 2: the delay after which the risk of dying from a radiation cancer disappears is increased by 5 years: $\omega_2 = 45$*

Once again the results are generally similar to those in the main study. The mean age at death for a radiation cancer victim is now up 1.2 years at 62.3 years when $T_R = 1$ year, reaches a maximum of 74.8 years, up 0.6 years, at $T_R = 85$ years, and then comes back to 63.6 years for an extremely long exposure, up 1.1 years.

Fig. 18 shows the life expectancy in the cohort of victims given a radiation exposure of $T_R$ years, their life expectancy if that radiation were to be completely averted and the difference between the two, which represents the loss of life expectancy for a given value of $T_R$. It is equivalent to Fig. 16 from the main study. The loss of life expectancy due to radiation exposure starts at 20.7 years, down 1.3 years, for a point exposure, reaches a minimum at $T_R = 85$ of 7.0 years, down 0.6 years, and then steadies out for an extremely long exposure at 17.4 years, a reduction of 1.1 years. The corresponding percentages of the loss of life expectancy caused by a fatal accident occurring when the radiation cancer victim is first exposed to radiation are 49.1%, 16,6% and 41.3%.

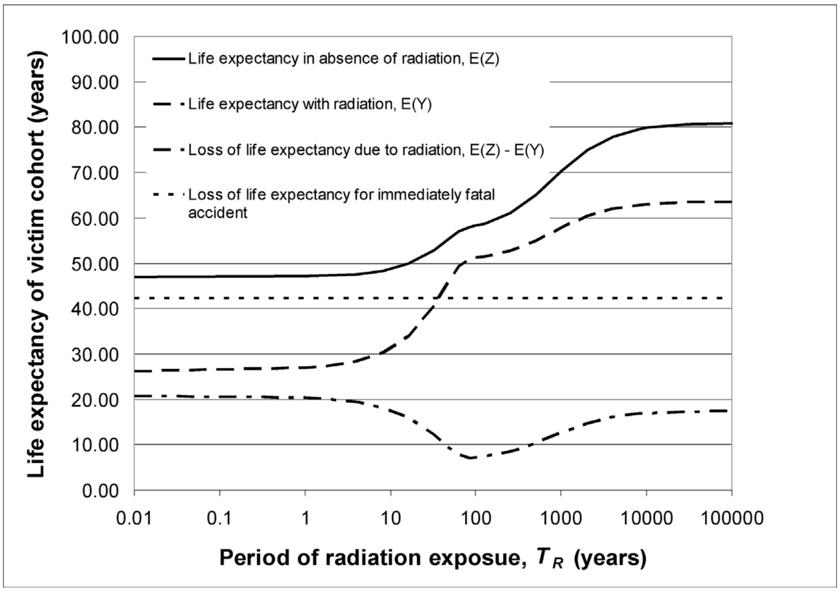


**Fig. 18 – Sensitivity Study 2: ω2 increased from 40 years to 45 years.**
**Life expectancy for cohort of radiation cancer victims exposed to radiation for a period, $T_R$ years; life expectancy for this cohort in the absence of radiation. Resultant loss of life expectancy due to radiation compared with the life expectancy from an immediately fatal accident.**

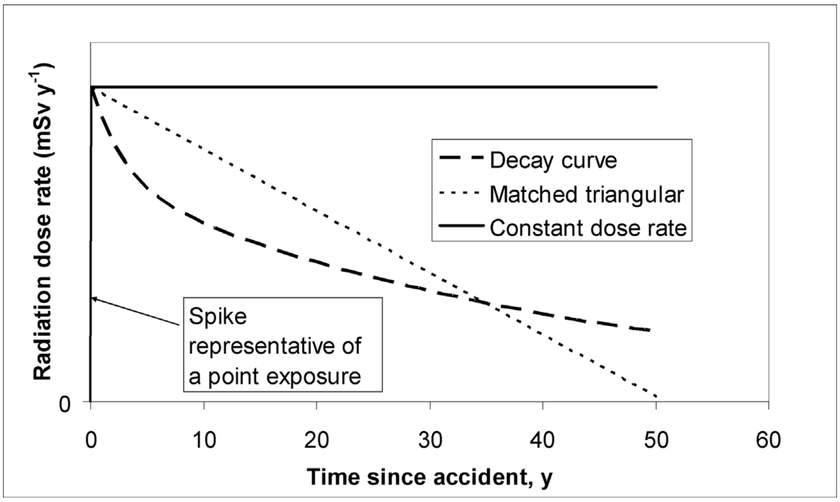


**Fig. 19 – Radiation dose profiles: a point dose, a constant dose rate, a decaying dose rate and a matched triangular dose rate.**

### *12.7. Applicability of the results to a large nuclear accident*

The two cases considered, point exposure and prolonged exposure at a constant annual rate, are immediately representative of a number of realistic situations. Moreover, the two cases will bracket a number of different dose profiles. For example a spike in radiation dose followed by a rapid decline over time is likely to resemble a point exposure in its effects, while the same spike followed by a slow decline can be expected to produce effects closer to a prolonged exposure at a constant annual rate.

The last mentioned case is characteristic of big nuclear accidents like Chernobyl and Fukushima Daiichi, and the argument for bracketing may be developed further as follows. Consider the generalised profile of radiation dose to a local population remaining in situ after a large nuclear release and deposition of radioactivity. The dose rate will decay over time as shown in Fig. 19, with the decline being dominated after a few years by the decay of caesium-137, which has a half-life of 30 years (in fact its effective half-life may be shorter than this due to dispersal away from man's environment).

The multiple exponential decay process may be approximated by a triangular decline from the same initial dose rate, as shown. Observe now that a triangular decline has two limiting curves associated with it:

(i) a vertical spike, approached as the slope is reduced from the finite negative value shown towards $-\infty$; and
(ii) a horizontal line, formed as the slope is increased from its finite negative value towards zero.

These limiting curves correspond to the point exposure and prolonged exposure at a constant annual rate mentioned in the first paragraph. These are the cases that have been explored in detail in this paper.

Note that, as discussed in Sections 12.1 and 12.2 above, the mean age at death for radiation cancer victims is similar whether they are subjected to a point radiation exposure (mean age at death = 60.6 years) or a constant continuing dose, especially when the continuing dose lasts less than 10 years (61–64 years) or is extremely long term (62.5 years). See Fig. 5. The maximum difference occurs when the continuing dose lasts for 85 years, at which period of exposure the mean age at death about 25% higher, at 74.2 years, than the point exposure figure.

Considerations of continuity suggest that the mean age at death for a triangular decline will lie somewhere between the figures characteristic of the two limiting cases, figures that are, in any case, rather similar. Since the triangular decline in dose is broadly representative of accident situations, it follows that the results presented in the paper may be regarded as characterising the age at death of radiation victims exposed after a big nuclear accident.

(With the triangular decline, the limiting forms of a point exposure and a continuing constant dose rate can be reached by adjusting only one parameter, namely the slope, between its limits $(-\infty, 0)$, and this makes the process easy to comprehend. But it is not strictly necessary to invoke the triangular decline, as the true radiation dose rate will fall in line with the equation, $d_r(t) = \sum_{i=1}^{n} \beta_i \exp(-t \ln 2/\tau_i)$, where $\tau_i$ is the half-life and $\beta_i$ a constant for the $i^{th}$ out of the $n$ radioactive species present. This expression too has the point exposure and the constant dose rate as limiting forms, but now it is necessary to adjust all the half-lives, $\tau_i$, $i = 1, 2, \ldots, n$, simultaneously between their limits $(0, \infty)$.)

## 13. Discussion

Scientists have analysed with considerable care data coming from cases of medical radiation and the Japanese Life Span Study of atomic bomb survivors, under the assumption that cancers of many different types may be induced by radiation. Nevertheless considerable uncertainties remain concerning the extent of human vulnerability across different populations to specific types of radiation induced cancers. Examples are the differences in susceptibility between atomic bomb survivors and radon exposed miners in the case of lung cancer and, more generally, between the Japanese and U.S. populations, particularly in the case of gastric cancer. See ICRP (2007a), Appendix A, paragraphs (A.110) to (A.140). These differences in vulnerability add to the overarching difficulty of resolving the effects of radiation doses below 100 mSv either per year or in a point exposure as a result of epidemiological analysis techniques running out of statistical power at low doses. In practice results at dose levels of several hundred mSv are used as the basis for an extrapolation to the lower dose levels, but, in the absence of hard evidence, that procedure relies on the use of judgement.

Each type of cancer will be subject to its own probability density for mortality period, and it is known that in all cases there will be a significant delay between radiation exposure and death for a radiation cancer victim. Moreover the data suggest that the risk of some form of radiation cancer will remain elevated for decades after exposure. In the absence of accurate disaggregated data on the probability distributions for mortality period for the different types of cancer, the Marshall team adopted the heuristic of assuming that the risk was uniform over a 30 year period lagged on exposure by 10 years. After extending the treatment so that it can cover prolonged radiation exposures, it has been found that the Marshall assumptions tally well with calculations carried out by Lochard and Schneider (1992), for members of the public exposed after Chernobyl (Waddington et al., 2017). Moreover, the sensitivity studies described in Sections 12.5 and 12.6 show that the paper's results are not altered significantly if the first radiation induced fatalities can occur after 5 rather than 10 years (setting $\omega_1 = 5$) or if the period of risk is extended out to 45 years after exposure by setting $\omega_2 = 45$ years.

The mean age at death from radiation cancers, between 61 and 75 years, found from the Marshall model and UK life table statistics (standard deviation of between 19 and 23 years), is comparable with the 73.5 years mean value (12.7 years standard deviation) calculated from figures for age at death from all cancers in the UK, 2010–2012 (Cancer Research UK, 2015), and also with the age at death from all cancers in California, 2000–2010: mean: 70.5, standard deviation: 14.4 years (California Department of Public Health, 2015). The degree of overlap will, of course, increase the difficulty of distinguishing radiation-induced cancers from the background level, especially when their number is small.

The cancer causing effects of nuclear radiation have been subject to intensive scientific scrutiny over a very long period, particularly since the atomic bombing of Hiroshima and Nagasaki 70 years ago. The time and resources applied internationally will surely have exceeded those devoted to the study of any other carcinogen, and it is likely that our understanding of the deleterious effects of nuclear radiation is correspondingly higher. While uncertainties remain, as discussed above, the Marshall probability density for radiation cancer mortality is proposed as a simple but robust tool for analysing the effects on the life expectancy of people exposed to nuclear radiation for safety and policy purposes.

Based on the Marshall model and the most up-to-date life tables, it is possible to say that if a representative sample of the UK population were subjected to a non-acute radiation dose, those who became radiation cancer victims would, on average, be aged over 60 when they died, whether the exposure occurred over a very short interval or a prolonged period. The reduction in life expectancy among the cohort of victims would always be less than 22 years, about half the loss, 42 years, associated with an immediately lethal accident such as a fatal car crash or a fatal train crash.

The last comparison is directly relevant under the view, held very widely in society and, indeed, enshrined in law, that life is preferred to death. It should be borne in mind, too, that the long latency period of radiation-induced cancers means that most of the two decades of extra life that the average radiation cancer victim gains over the fatal car crash or rail crash victim can be expected to be lived normally, with living becoming restricted, with an attendant eventual need for palliative care, only near the end of the period.

[Those favouring the human capital approach, which values people's lives only on their net economic benefit to society, would evaluate the radiation cancer victim's economic contribution during his/her extra productive years and subtract the care costs incurred in the final few years. Given the very large disparity in survival times between the two forms of premature death, the notional monetary balance would almost certainly be positive in favour of death from a radiation cancer. But the human capital concept has been largely abandoned by the UK Government which has now adopted a "willingness to pay" approach, although its chosen replacement vehicle, the "value of a prevented fatality" (VPF) is based on a method of estimation that contains such gross flaws that no reliance can be placed on it (see Thomas and Waddington, 2017b, and, for more detail, Thomas and Vaughan, 2015a,b,c). Ethically superior approaches recognise the number of life-years lost as the true measure of the impact of an injury or illness leading to premature death, as opposed to assigning a uniform value to every prevented fatality (Nathwani and Lind, 1997; Nathwani et al. 2009; Sunstein, 2003; Broome, 2006; Thomas et al., 2006, 2010). See Thomas and Vaughan (2013) for a further discussion of the philosophy of valuing people's lives.]

The average age at death will rise as the length of the radiation exposure increases before peaking at 74 years for an exposure of 85 years, and then falling back to 62.5 years as the period of exposure increases to extremely long values. While there will be more radiation cancer victims at longer exposures, the extended period of exposure will mean that the risk to people of a given starting age will persist for longer, allowing more victims to die at a higher age. Victims will thus tend to be older when they die, pushing up the average age at death. But as the length of exposure increases still further, more and more people will be born into the exposure, causing the weighting given to those with a zero starting age to rise. While this group has the lowest life expectancy from exposure to radiation, even so its life expectancy is still high (62.5 years) for exposures lasting 91 years or more. The figure of 62.5 years will come to dominate the age at death averaged across the inclusive cohort as the period of exposure becomes extremely long.

As shown by the probability densities in Figs. 6–9, the probability of dying at a young age cannot be ruled out, so it is not surprising that the spread of ages at death for radiation cancer victims can be quite large. The effect may be quantified using a 67% survival age, which represents a chance of living longer that is more than double that of dying younger. See Fig. 5. Two thirds of the cohort can be expected to live to 48 years or older, irrespective of the length of exposure. The 67% survival age rises to more than 65 years for a long-term exposure continuing for between 60 and 220 years.

The paper's results assume a risk coefficient with the general properties promulgated by the ICRP, viz. a common figure for all members of the public and no lower threshold for radiation risk. However they are independent of the exact size of the risk coefficient and apply to any dose rate where acute effects are not expected. Thus the finding that radiation cancer victims are likely, on average, to live into their seventh or eighth decade emerges as robust against possible re-evaluations of the risk coefficient in the future. Their reduction in life expectancy will be less than about a half what they would have lost if, instead, they had been involved in an immediately lethal accident such as a fatal car crash or a major accident on the railway.

On average, therefore, a UK citizen has twice as much or more to lose from being killed outright in a road or rail accident as opposed to dying as result of a radiation-induced cancer. This illustrates the clear limitations inherent in the UK Government's current approach whereby a one-size-fits-all "value of a prevented fatality" (VPF) is applied in cost-benefit analysis when the life expectancy at stake can be very different. This problem with applying the VPF in the context of radiological protection is additional to the gross flaws previously uncovered in the value assigned to the VPF in the UK.

Interestingly, the error is compounded by the Health and Safety Executive (2001, 2015), which, in its document, "Reducing risk, protecting people", states

> "Currently, HSE takes the view that it is only in the case where death is caused by cancer that people are prepared to pay a premium for the benefit of preventing a fatality and has accordingly adopted a VPF twice that of the roads benchmark figure." (Appendix C, paragraph 13.)

Unfortunately, as noted above, the "roads benchmark" VPF cited also lacks validity, having been shown to be based on a study that was too flawed to be used (Thomas and Vaughan, 2015a,b,c).

## 14. Conclusions

The paper treats the question of the loss of life expectancy amongst radiation cancer victims in a general way. The results are applicable inter alia to the radiation doses received by the public after a large accident such as Chernobyl and Fukushima Daiichi.

It is found that the average radiation cancer victim will live into his or her 60s or 70s, depending on how long the radiation exposure lasts, based on data from the UK life tables. Meanwhile the reduction in life expectancy among the cohort of victims would always be less than 22 years, and may be as low as 8 years.

The probability distribution for age at death has been quantified, and this has allowed the calculation of the 67% survival age for the victims, where the chance of living longer is more than double that of dying younger. This is always at least 48 years and can rise to 66 years, depending on the duration of the exposure, assumed to be constant year on year.

An important feature of the results presented is that they apply to any exposure to radiation between a point dose and a constant annual dose that does not cause radiation sickness. The figures presented, for both point and constant annual exposures, are equally valid whether the dose is a few mSv or a few hundred mSv. Nor is the outcome affected by the magnitude of the coefficient used to convert radiation dose into risk.

On average a UK citizen has at least twice as much to lose from being killed outright in a road or rail accident as opposed to dying as result of a radiation-induced cancer. This result calls into question once again the concept of the Value of a Prevented Fatality still used for cost-benefit analyses in the UK on a "one size fits all" basis, which disregards the amount of life expectancy lost. This problem with applying the VPF in the context of radiological protection is additional to the gross flaws previously uncovered in the value assigned to the VPF in the UK. It is clear that the VPF should not be used as a criterion for cost-benefit analysis in radiological protection.

The results may reinforce the notion that nuclear radiation causes less damage than many might have believed.

## Acknowledgements

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The data used in the paper are freely available at the website of the Office of National Statistics cited in the text.

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## Appendix A. Introduction to step or jump functions, $J_p(.)$

This Appendix explores the calculus of step or jump functions, $J_p(.)$, so as to generate results to be used in the main text. First, starting with the integral of a jump function that steps up after a general time interval, $\omega$:

$$\int_{m=0}^{z} J_p(m-\omega)\,dm = \left\{ \begin{array}{ll} \int_{m=0}^{z} 0dm = 0 & \text{for } z \le \omega\text{, implying all } m \le \omega \\ \int_{m=\omega}^{z} dm = z-\omega & \text{for } z > \omega \end{array} \right\} = (z-\omega)J_p(z-\omega) \tag{A.1}$$

It will also useful to integrate the result of Eq. (A.1):

$$\int_{m=0}^{z} (m-\omega)J_p(m-\omega)\,dm = \left\{ \begin{array}{ll} \int_{m=0}^{z} 0dm = 0 & \text{for } z \le \omega \\ \int_{m=\omega}^{z} (m-\omega)\,dm = \frac{1}{2}(z-\omega)^2 & \text{for } z > \omega \end{array} \right\} = \frac{1}{2}(z-\omega)^2 J_p(z-\omega) \tag{A.2}$$

Hence we may define a further function, $\psi_1(z)$, as the definite integral from 0 to z of the function, $\psi_0(m)$, given by Eq. (24) in the main text:

$$\begin{aligned} \psi_1(z) &= \int_{m=0}^{z} \psi_0(m)\,dm = \int_{m=0}^{z} (m-\omega_1)J_p(m-\omega_1)\,dm - \int_{m=0}^{z} (m-\omega_2)J_p(m-\omega_2)\,dm - \int_{m=0}^{z} (m-\omega_3)J_p(m-\omega_3)\,dm \\ &+ \int_{m=0}^{z} (m-\omega_4)J_p(m-\omega_4)\,dm = \frac{1}{2}\left\{(z-\omega_1)^2 J_p(z-\omega_1) - (z-\omega_2)^2 J_p(z-\omega_2)\right\} - \frac{1}{2}\left\{(z-\omega_3)^2 J_p(z-\omega_3) - (z-\omega_4)^2 J_p(z-\omega_4)\right\} \end{aligned} \tag{A.3}$$

In addition, it is useful to show that

$$\int_{m=0}^{z} m(m-\omega)J_p(m-\omega)\,dm = \left\{ \begin{array}{ll} \int_{m=0}^{z} 0dm = 0 & \text{for } z \le \omega \\ \left\{ \begin{array}{l} \int_{m=\omega}^{z} m^2 dm - \omega \int_{m=\omega}^{z} mdm \\ = \frac{1}{3}\left(z^3-\omega^3\right) - \frac{\omega}{2}\left(z^2-\omega^2\right) \end{array} \right\} & \text{for } z > \omega \end{array} \right\} = \frac{1}{6}\left(2\left(z^3-\omega^3\right) - 3\omega\left(z^2-\omega^2\right)\right) J_p(z-\omega) \tag{A.4}$$

Hence it is possible to define a further function, $\theta_1(z)$, by

$$\theta_1(z) = \int_{m=0}^{z} m\psi_0(m)\,dm = \int_{m=0}^{z} m(m-\omega_1)J_p(m-\omega_1)\,dm - \int_{m=0}^{z} m(m-\omega_2)J_p(m-\omega_2)\,dm - \int_{m=0}^{z} m(m-\omega_3)J_p(m-\omega_3)\,dm$$
$$+\int_{m=0}^{z} m(m-\omega_4)J_p(m-\omega_4)\,dm = \frac{1}{6}\left\{\left(2\left(z^3-\omega_1^3\right)-3\omega_1\left(z^2-\omega_1^2\right)\right)J_p(z-\omega_1)-\left(2\left(z^3-\omega_2^3\right)-3\omega_2\left(z^2-\omega_2^2\right)\right)J_p(z-\omega_2)\right\}$$
$$-\frac{1}{6}\left\{\left(2\left(z^3-\omega_3^3\right)-3\omega_3\left(z^2-\omega_3^2\right)\right)J_p(z-\omega_3)-\left(2\left(z^3-\omega_4^3\right)-3\omega_4\left(z^2-\omega_4^2\right)\right)J_p(z-\omega_4)\right\} \quad \text{(A.5)}$$

It is also useful to consider

$$\int_{m=0}^{z} m^2(m-\omega)J_p(m-\omega)\,dm = \begin{cases} \int_{m=0}^{z} 0dm = 0 & \text{for } z \le \omega \\ \int_{m=\omega}^{z} m^3dm - \omega\int_{m=\omega}^{z} m^2dm = \frac{1}{4}\left(z^4-\omega^4\right)-\frac{\omega}{3}\left(z^3-\omega^3\right) & \text{for } z > \omega \end{cases}$$
$$= \frac{1}{12}\left(3\left(z^4-\omega^4\right)-4\omega\left(z^3-\omega^3\right)\right)J_p(z-\omega) \quad \text{(A.6)}$$

Thus the function, $\theta_2(z)$, may be found:

$$\theta_2(z) = \int_{m=0}^{z} m^2\psi_0(m)\,dm = \int_{m=0}^{z} m^2(m-\omega_1)J_p(m-\omega_1)\,dm - \int_{m=0}^{z} m^2(m-\omega_2)J_p(m-\omega_2)\,dm - \int_{m=0}^{z} m^2(m-\omega_3)J_p(m-\omega_3)\,dm$$
$$+\int_{m=0}^{z} m^2(m-\omega_4)J_p(m-\omega_4)\,dm = \frac{1}{12}\left\{\left(3\left(z^4-\omega_1^4\right)-4\omega_1\left(z^3-\omega_1^3\right)\right)J_p(z-\omega_1)-\left(3\left(z^4-\omega_2^4\right)-4\omega_2\left(z^3-\omega_2^3\right)\right)J_p(z-\omega_2)\right\}$$
$$-\frac{1}{12}\left\{\left(3\left(z^4-\omega_3^4\right)-4\omega_3\left(z^3-\omega_3^3\right)\right)J_p(z-\omega_3)-\left(3\left(z^4-\omega_4^4\right)-4\omega_4\left(z^3-\omega_4^3\right)\right)J_p(z-\omega_4)\right\} \quad \text{(A.7)}$$

## Appendix B. Probability distribution for age in a steady-state population

Consider a population in a steady state, which implies that the number of people born each year will be constant and that the actuarial parameters will be unchanging with time. Let $n(a)$ ($y^{-1}$) be the population density at age, $a$, implying that the number of people between ages $a$ and $a+da$ is $n(a)da$. The number $n(a)$ may also be regarded as the rate at which people are reaching age, $a$, and will be the same as the number of people born $a$ years before, in the interval 0 to $da$, multiplied by the probability, $S(a)$, of surviving to age, $a$, namely $n(0)S(a)da$

$$n(a)\,da = n(0)\,S(a)\,da \quad \text{(B.1)}$$

where $n(0)$ is the birth rate ($y^{-1}$).

We may integrate (B.1) between ages 0 and infinity:

$$\int_0^{\infty} n(a)\,da = N = n(0)\int_0^{\infty} S(a)\,da \quad \text{(B.2)}$$

where $N$ is the total number in the population across all ages. Noting that

$$\int_0^{\infty} S(a)\,dt = X(0) \quad \text{(B.3)}$$

where $X(0)$ is the life-expectancy of an individual at birth, it follows that the density of people being born (the rate at which people are being born or birth-rate) in a steady-state population is:

$$n(0) = \frac{N}{X(0)} \tag{B.4}$$

The density of people aged $a$ may be found by combining (B.4) with (B.1):

$$n(a) = \frac{S(a)}{X(0)} N \tag{B.5}$$

The probability density for age, $a$, in a steady-state population is found by dividing both sides by the total population, $N$:

$$f_A(a) = \frac{n(a)}{N} = \frac{S(a)}{X(0)} \tag{B.6}$$

Since the survival probability at age 0 is unity, we may deduce from Eq. (B.6) that

$$f_A(0) = \frac{S(0)}{X(0)} = \frac{1}{X(0)} \tag{B.7}$$

## Appendix C. The properties of the integral $\int_{b=0}^{T_R} \frac{\psi_1^{(T_R-b)}(\alpha_0)}{T_R-b} db$

Put $u = T_R - b$, so that $du = -db$, and $u = T_R$ when $b = 0$, while $u = 0$ when $b = T_R$. This allows the integral to be recast as:

$$\int_{b=0}^{T_R} \frac{\psi_1^{(T_R-b)}(\alpha_0)}{T_R - b} db = -\int_{u=T_R}^{0} \frac{\psi_1^{(u)}(\alpha_0)}{u} du = \int_{u=0}^{T_R} \frac{\psi_1^{(u)}(\alpha_0)}{u} du \tag{C.1}$$

where $\psi_1^{(u)}$ is a generalisation of $\psi_1^{(T_R)}(z) = \psi_1(z)$, as explained under Eq. (31) in the main text. Thus substituting $u$ for $T_R$ in Eqs. (25) and (26) and the resultant expressions for $\omega_3$ and $\omega_4$ in Eq. (A.3) gives

$$\psi_1^{(u)}(\alpha_0) = \frac{1}{2}\left\{(\alpha_0 - \omega_1)^2 J_p(\alpha_0 - \omega_1) - (\alpha_0 - \omega_2)^2 J_p(\alpha_0 - \omega_2)\right\} - \frac{1}{2}\left\{(\alpha_0 - \omega_1 - u)^2 J_p(\alpha_0 - \omega_1 - u) - (\alpha_0 - \omega_2 - u)^2 J_p(\alpha_0 - \omega_2 - u)\right\} \tag{C.2}$$

where Eq. (25) and (26) have been used to substitute for $\omega_3$ and $\omega_4$. Since $\alpha_0 = 101$, $\omega_1 = 10$ and $\omega_2 = 40$, $\alpha_0 > \omega_2 > \omega_1$ so that the first two jump functions will always have a positive argument and hence return 1.0. Thus Eq. (C.2) may be simplified to

$$\psi_1^{(u)}(\alpha_0) = \frac{(\alpha_0 - \omega_1)^2 - (\alpha_0 - \omega_2)^2}{2} - \frac{1}{2}\left\{(\alpha_0 - \omega_1 - u)^2 J_p(\alpha_0 - \omega_1 - u) - (\alpha_0 - \omega_2 - u)^2 J_p(\alpha_0 - \omega_2 - u)\right\} \tag{C.3}$$

The jump function $J_p(\alpha_0 - \omega_2 - u)$ will return a value of 1.0 when $\alpha_0 \geq \omega_2 + u$. Moreover, $\alpha_0 \geq \omega_2 + u$ implies $\alpha_0 \geq \omega_1 + u$, since $\omega_1 < \omega_2$, so that $J_p(\alpha_0 - \omega_1 - u)$ will return a value of 1.0 also. Hence we may write Eq. (C.3) as:

$$\begin{aligned}
\psi_1^{(u)}(\alpha_0) &= \frac{1}{2}\left((\alpha_0 - \omega_1)^2 - (\alpha_0 - \omega_1)^2 - \left\{\begin{array}{l}(\alpha_0 - \omega_1 - u)^2 \\ -(\alpha_0 - \omega_2 - u)^2\end{array}\right\}\right) \\
&= \frac{1}{2}\left(\begin{array}{l}(\alpha_0 - \omega_1)^2 - (\alpha_0 - \omega_1)^2 \\ -\left\{\begin{array}{l}(\alpha_0 - \omega_1)^2 - 2u(\alpha_0 - \omega_1) + u^2 \\ -\left((\alpha_0 - \omega_2)^2 - 2u(\alpha_0 - \omega_2) + u^2\right)\end{array}\right\}\end{array}\right) \quad \text{for} \quad u \leq \alpha_0 - \omega_2 \\
&= u(\alpha_0 - \omega_1) - u(\alpha_0 - \omega_2) = u(\omega_2 - \omega_1) \\
&= u\Omega
\end{aligned} \tag{C.4}$$

Thus

$$\frac{\psi_1^{(u)}(\alpha_0)}{u} = \Omega \quad \text{for} \quad u \leq \alpha_0 - \omega_2 \tag{C.5}$$

Moving on to consider the range of $u$ above $\alpha_0-\omega_2$ but no greater than $\alpha_0-\omega_1$: now $J_p(\alpha_0-\omega_2-u)=0$ because $u>\alpha_0-\omega_2$, while $J_p(\alpha_0-\omega_1-u)=1$ since $u\le\alpha_0-\omega_1$. Hence Eq. (C.3) becomes:

$$\begin{aligned}\psi_1^{(u)}(\alpha_0)&=\frac{1}{2}\left((\alpha_0-\omega_1)^2-(\alpha_0-\omega_2)^2-(\alpha_0-\omega_1-u)^2\right)\\&=\frac{1}{2}\begin{pmatrix}(\alpha_0-\omega_1)^2-(\alpha_0-\omega_2)^2\\-\left((\alpha_0-\omega_1)^2-2u(\alpha_0-\omega_1)+u^2\right)\end{pmatrix}\qquad\text{for}\quad \alpha_0-\omega_2<u\le\alpha_0-\omega_1\\&=u(\alpha_0-\omega_1)-\frac{u^2}{2}-\frac{(\alpha_0-\omega_2)^2}{2}\end{aligned}\tag{C.6}$$

so that

$$\frac{\psi_1^{(u)}(\alpha_0)}{u}=(\alpha_0-\omega_1)-\frac{u}{2}-\frac{(\alpha_0-\omega_2)^2}{2u}\quad\text{for }\alpha_0-\omega_2<u\le\alpha_0-\omega_1\tag{C.7}$$

In the cases where $u>\alpha_0-\omega_1$ both jump functions return 0.0 and so Eq. (C.3) takes the simple form

$$\psi_1^{(u)}(\alpha_0)=\frac{(\alpha_0-\omega_1)^2-(\alpha_0-\omega_2)^2}{2}\quad\text{for }u>\alpha_0-\omega_1\tag{C.8}$$

and

$$\frac{\psi_1^{(u)}(\alpha_0)}{u}=\frac{(\alpha_0-\omega_1)^2-(\alpha_0-\omega_2)^2}{2u}\quad\text{for}\quad u>\alpha_0-\omega_1\tag{C.9}$$

Since $\max_b u=\max_b(T_R-b)=T_R$, the value of $T_R$ decides the finishing point for $u$ in the integral, $\int_{u=0}^{T_R}\frac{\psi_1^{(u)}(\alpha_0)}{u}du$, which will be

(i) no greater than $\alpha_0-\omega_2$ when $T_R\le\alpha_0-\omega_2$
(ii) above $\alpha_0-\omega_2$ but no greater than $\alpha_0-\omega_1$, when $\alpha_0-\omega_2<T_R\le\alpha_0-\omega_1$
(iii) above $\alpha_0-\omega_1$ when $T_R>\alpha_0-\omega_1$.

The solution to the integral will take different forms depending on which of the three conditions (i), (ii) or (iii) pertains.

(i) When $T_R\le\alpha_0-\omega_2$

$$\int_{u=0}^{T_R}\frac{\psi_1^{(u)}(\alpha_0)}{u}du=\Omega T_R\tag{C.10}$$

(ii) When $\alpha_0-\omega_2<T_R\le\alpha_0-\omega_1$

We note now that

$$\int_{u=0}^{T_R}\frac{\psi_1^{(u)}(\alpha_0)}{u}du=\int_{u=0}^{\alpha_0-\omega_2}\frac{\psi_1^{(u)}(\alpha_0)}{u}du+\int_{u=\alpha_0-\omega_2}^{T_R}\frac{\psi_1^{(u)}(\alpha_0)}{u}du\tag{C.11}$$

Integrating Eq. (C.5) between 0 and $\alpha_0-\omega_2$:

$$\int_{u=0}^{\alpha_0-\omega_2}\frac{\psi_1^{(u)}(\alpha_0)}{u}du=\Omega(\alpha_0-\omega_2)\tag{C.12}$$

Turning to the second integral on the right hand side of Eq. (C.11):

$$\begin{aligned}\int_{u=\alpha_0-\omega_2}^{T_R}\frac{\psi_1^{(u)}(\alpha_0)}{u}du&=\int_{u=\alpha_0-\omega_2}^{T_R}\left((\alpha_0-\omega_1)-\frac{u}{2}-\frac{(\alpha_0-\omega_2)^2}{2u}\right)du=(\alpha_0-\omega_1)[u]_{\alpha_0-\omega_2}^{T_R}-\frac{1}{4}\left[u^2\right]_{\alpha_0-\omega_2}^{T_R}-\frac{(\alpha_0-\omega_2)^2}{2}[\ln u]_{\alpha_0-\omega_2}^{T_R}\\&=(\alpha_0-\omega_1)(T_R-\alpha_0+\omega_2)-\frac{T_R^2-(\alpha_0-\omega_2)^2}{4}-\frac{(\alpha_0-\omega_2)^2}{2}\ln\left(\frac{T_R}{\alpha_0-\omega_2}\right)=\frac{(T_R-\alpha_0+\omega_2)(4\alpha_0-4\omega_1)-(T_R-\alpha_0+\omega_2)(T_R+\alpha_0-\omega_2)}{4}\\&\quad-\frac{(\alpha_0-\omega_2)^2}{2}\ln\left(\frac{T_R}{\alpha_0-\omega_2}\right)=\frac{(T_R-(\alpha_0-\omega_2))(3(\alpha_0-\omega_1)+\Omega-T_R)}{4}-\frac{(\alpha_0-\omega_2)^2}{2}\ln\left(\frac{T_R}{\alpha_0-\omega_2}\right)\end{aligned}\tag{C.13}$$

Substituting from Eqs. (C.12) and (C.13) into Eq. (C.11) gives:

$$\int_{u=0}^{T_R} \frac{\psi_1^{(u)}(\alpha_0)}{u} du = \Omega(\alpha_0 - \omega_2) + \frac{1}{4}(T_R - (\alpha_0 - \omega_2))(3(\alpha_0 - \omega_1) + \Omega - T_R) - \frac{(\alpha_0 - \omega_2)^2}{2} \ln\left(\frac{T_R}{\alpha_0 - \omega_2}\right) \tag{C.14}$$

(iii) When $T_R > \alpha_0 - \omega_1$

We now expand the integral as:

$$\int_{u=0}^{T_R} \frac{\psi_1^{(u)}(\alpha_0)}{u} du = \int_{u=0}^{\alpha_0-\omega_2} \frac{\psi_1^{(u)}(\alpha_0)}{u} du + \int_{u=\alpha_0-\omega_2}^{\alpha_0-\omega_1} \frac{\psi_1^{(u)}(\alpha_0)}{u} du + \int_{u=\alpha_0-\omega_1}^{T_R} \frac{\psi_1^{(u)}(\alpha_0)}{u} du \tag{C.15}$$

Meanwhile, substituting $\alpha_0 - \omega_1$ in place of $T_R$ in Eq. (C.13) gives:

$$\begin{aligned} \int_{u=\alpha_0-\omega_2}^{\alpha_0-\omega_1} \frac{\psi_1^{(u)}(\alpha_0)}{u} du &= (\alpha_0 - \omega_1)^2 - (\alpha_0 - \omega_1)(\alpha_0 - \omega_2) - \frac{(\alpha_0 - \omega_1)^2 - (\alpha_0 - \omega_2)^2}{4} - \frac{(\alpha_0 - \omega_2)^2}{2} \ln \frac{\alpha_0 - \omega_1}{\alpha_0 - \omega_2} = (\alpha_0 - \omega_1)(\omega_2 - \omega_1) \\ &- \frac{1}{4}(\omega_2 - \omega_1)(2\alpha_0 - \omega_1 - \omega_2) - \frac{(\alpha_0 - \omega_2)^2}{2} \ln \frac{\alpha_0 - \omega_1}{\alpha_0 - \omega_2} = \frac{\Omega}{4}(2\alpha_0 - 3\omega_1 + \omega_2) - \frac{(\alpha_0 - \omega_2)^2}{2} \ln \frac{\alpha_0 - \omega_1}{\alpha_0 - \omega_2} \end{aligned} \tag{C.16}$$

Integrating Eq. (C.9) between $\alpha_0 - \omega_2$ and $T_R$:

$$\begin{aligned} \int_{u=\alpha_0-\omega_1}^{T_R} \frac{\psi_1^{(u)}(\alpha_0)}{u} du &= \int_{u=\alpha_0-\omega_1}^{T_R} \frac{(\alpha_0 - \omega_1)^2 - (\alpha_0 - \omega_2)^2}{2u} du = \frac{(\alpha_0 - \omega_1)^2 - (\alpha_0 - \omega_2)^2}{2} \ln\left(\frac{T_R}{\alpha_0 - \omega_1}\right) = \frac{\omega_2 - \omega_1}{2}(2\alpha_0 - (\omega_1 + \omega_2)) \\ &\times \ln\left(\frac{T_R}{\alpha_0 - \omega_1}\right) = \Omega\left(\alpha_0 - \frac{\omega_1 + \omega_2}{2}\right) \ln\left(\frac{T_R}{\alpha_0 - \omega_1}\right) \end{aligned} \tag{C.17}$$

Substituting from Eqs. (C.12), (C.16) and (C.17) gives the complete integral:

$$\int_{u=0}^{T_R} \frac{\psi_1^{(u)}(\alpha_0)}{u} du = \Omega(\alpha_0 - \omega_2) + \frac{\Omega}{4}(2\alpha_0 - 3\omega_1 + \omega_2) - \frac{(\alpha_0 - \omega_2)^2}{2} \ln \frac{\alpha_0 - \omega_1}{\alpha_0 - \omega_2} + \Omega\left(\alpha_0 - \frac{\omega_1 + \omega_2}{2}\right) \ln\left(\frac{T_R}{\alpha_0 - \omega_1}\right) \tag{C.18}$$

## Appendix D. Summary of the main assumptions and definitions

1 The Marshall probability density for the time between radiation exposure and death from a radiation-induced cancer is assumed valid, as given in Fig. 3. The base-line values for time lags, $\omega_1$ and $\omega_2$, are 10 years and 40 years.
Grounds for the Marshall model constituting a reasonable and useful description of reality are provided in Section 1, Introduction.
Sensitivity studies are carried out for the cases (i) where $\omega_1$ is decreased by 5 years to 5 years and (ii) where $\omega_2$ is increased by 5 years to 45 years.

2 A "radiation cancer victim" is defined for the specific purposes of this paper as an individual who will die prematurely as a result of contracting a cancer induced by exposure to ionising radiation at some previous time.
Note that, by this definition, someone who contracts a radiation-induced cancer but does not die as a result of it, while undoubtedly an injured party, is not classed as a radiation cancer victim and may be called a radiation cancer survivor.

3 The "starting age" for a radiation cancer victim is his or her age when the radiation exposure starts.
4 No person and specifically no radiation cancer victim will live past the age, $\alpha_0$, where $\alpha_0$ has been taken to be 101, in line with UK life tables.
5 The probability of persons of all ages and both genders contracting a fatal cancer is assumed proportional to the radiation dose received provided the dose is below the level where acute effects (radiation sickness) are expected, so that only stochastic effects are anticipated. The risk coefficient is assumed the same for all, and no lower threshold for harm is assumed.
This is in line with the provision by the International Commission on Radiological Protection (ICRP) of a single risk coefficient for members of the public (both genders and all ages). The ICRP practice of assuming no lower threshold for radiation risk is also followed.
Experience has shown that the general public is likely only ever to be subject to stochastic effects of radiation even after a major nuclear accident. At neither of the big nuclear accidents at Chernobyl and Fukushima Daiichi was a member of the public subjected to a dose level leading to acute effects, which are expected only when a dose of about 500 mSv or more is received over a short period. A few percent of the 116,000 people moved away in 1986 as part of the first Chernobyl relocation

would have received a dose of roughly this level over a 12 month period if they had stayed in situ. No member of the public would have received such a dose at Fukushima Daiichi if no-one had been relocated. See Waddington et al. (2017).

6 The radiation dose is assumed to be below the level where acute effects are expected
7 No specific figure need be nor has been assumed for the risk coefficient for fatal cancer from nuclear radiation.
8 No specific figure for the size of the point dose or the continuing annual dose need be nor has been assumed.
9 The population exposed to radiation is assumed to be in a steady state, which implies that, although its members are being replaced continually, the number of people making up the population will be the same at all times. Under the steady-state condition, equal numbers are being born and dying each year, and this means that the size of the population will remain constant over time.
The steady-state idealisation brings mathematical simplification while retaining approximate validity for the national populations of many developed countries. It may be applied to population subsets also, such as a town or a large village, to which the actuarial statistics of the containing nation may be taken to apply. The mathematics associated with the assumption of a steady-state population are described in Appendix B.

10 Perturbations to the individual's hazard rate due to radiation exposure are assumed to take place from the date of birth.
This assumption allows direct use to be made of the actuarial life tables currently provided in the UK by the Office of National Statistics. The ICRP judges the in-utero risk "to be no greater than the following exposure in early childhood" and provides no specific in-utero risk coefficient (ICRP, 2007c, paragraph A.171). The sensitivity study that brought the earliest onset of mortality forward by 5 years caused the mean age at death from a radiation-induced cancer to decrease by only 1 or 2 years. This effect is likely to be larger than that would be produced taking the age of first vulnerability back by about ¾ year.
11 Two forms of dose profile are used: (i) a constant dose rate for a variable number of years and (ii) a dose delivered at a point in time.
The dose profiles (i) and (ii) bracket the type of dose profile likely to be experienced as a result of nuclear fallout after a big nuclear accident. The age at death from a radiation-induced cancer as a result of a big nuclear accident can be expected to be similarly bracketed by the ages at death associated with profiles (i) and (ii). Arguments for this conclusion are given in Section 12.7.
12 The paper makes use of the National Life Tables for 2012–2014, provided for the UK by the Office of National Statistics (Office for National Statistics, 2015).
The tables take account of all registered births and deaths in the UK and cover a 3-year rolling period to reduce the effect of annual fluctuations in the number of deaths caused by seasonal events such as flu. The figures provided are based on the age-specific mortality rates of the given area and time period. The life expectancy projections are idealised in that they are made on the basis that the mortality rates do not vary. In fact there has been a continuing trend towards increased life expectancies over the past 50 years.

## Appendix E. Nomenclature

| | |
|---|---|
| $a$ | Age (y) |
| $a$, $A$ | Age at which the radiation dose starts, known as "starting age". A capital letter used when the starting age is assumed to be a random variable (y) |
| $b$ | Time after the onset of the radiation dose before the person is born (y) |
| $c_v(a)$ | Fraction of the people of age, $a$, at the start of the exposure who will become radiation cancer victims |
| $c_v$ | Fraction of people of any given age at the start of the exposure who will become radiation cancer victims |
| $d_a$ | Constant dose rate (Sv y$^{-1}$) |
| $d_{rx}(x)$ | Dose rate experienced at age, $x$ (Sv y$^{-1}$) |
| $d_{rt}(t)$ | Dose rate experienced at time, $t$ (Sv y$^{-1}$) |
| $D_h(a)$ | Maximum integrated dose that can cause harm to a person of starting age, $a$ (Sv) |
| $f_A(a)$ | Probability density for age, $a$ (y$^{-1}$) |
| $f_M(m)$ | Probability density for mortality period, $m$ (y$^{-1}$) |
| $f_{vA}(a)$ | Probability density for starting age, $a$, for a radiation cancer victim (y$^{-1}$) |
| $f_Y(y)$ | Probability density for age at death, $y$, for a radiation cancer victim (y$^{-1}$) |
| $f_{Y\|A}(y\|a)$ | Probability density for age at death, $y$, for a radiation cancer victim, given starting age, $a$ (y$^{-1}$) |
| $f^*_{Y\|A}(y\|a)$ | Probability density for age at death, $y$, for a radiation cancer victim, given starting age, $a$, valid over a limited range of starting ages (y$^{-1}$) |
| $J_p(.)$ | Step or "jump" function |
| $L$ | Life to come (y) |
| $L^*$ | Life to come in the absence of radiation exposure (y) |
| $m$ | Number of people becoming radiation cancer victims as a result of a radiation exposure of duration, $T_R$. Used in Section 12.4 |
| $m$, $M$ | Mortality period: the time between cancer induction and death for a radiation cancer victim. A capital letter is used when the mortality period is regarded as a random variable (y) |
| $N$ | Total number of people in the population over all ages |

| | |
|---|---|
| $n$ | Number of standard deviations (need not be an integer). Used in Section 11 |
| $n(a)$ | Population density at age, $a$ |
| $N(a)$ | Number of people of age, $a$ |
| $n_2$ | Number of people born into the population between $b$ and $b+1$ years after the onset of radiation exposure (y) |
| $n_{2v}^{(b)}$ | Density of radiation cancer victims born $b$ years after the start of the radiation exposure ($y^{-1}$) |
| $N_v$ | Number of radiation cancer victims over all ages |
| $N_v(a)$ | Number of radiation cancer victims of age, $a$ |
| $N_{v0}(a)$ | Estimate of the number of radiation cancer victims of age, $a$, when no upper limit is set on age at death |
| $q_X(x)$ | Probability density for cancer induction at age, $x$, for a radiation cancer victim ($y^{-1}$) |
| $q_{X\|A}(x\|a)$ | Probability density for cancer induction at age, $x$, for a radiation cancer victim, given starting age, $a$ ($y^{-1}$) |
| $S(a)$ | Probability of surviving to age, $a$ |
| $t$ | Time (y) |
| $T$ | Randomly chosen time (y) |
| $T_R$ | Duration of radiation exposure (y) |
| $T_R^{(b)}$ | Effective duration of radiation exposure for someone born $b$ years after the start of the exposure: $T_R^{(b)} = T_R - b$ (y) |
| $V$ | General random variable used in Section 11 |
| $x, X$ | Age of an individual; age at which the cancer is induced for a radiation cancer victim. A capital letter is used when the age is assumed to be a random variable (y) |
| $x(a)$ | Life expectancy at age, $a$, as given by the life tables (y) |
| $X(0)$ | Life expectancy at birth (age, 0) (y) |
| $y, Y$ | Age at death for a radiation cancer victim. A capital letter is used when the age at death is assumed to be a random variable (y) |
| $y_s$ | Selected value of age at death from a radiation cancer (y) |
| $z$ | General variable |
| $\alpha_0$ | Limiting maximum age at which death from extreme old age will occur (y) |
| $\beta_0$ | A specified value of age (y) |
| $\beta_i$ | A constant characterising the radiation dose rate for the $i^{th}$ radioactive species ($Sv\,y^{-1}\,kg^{-1}$) |
| $\theta_1(.)$ | Jump function expression defined by Eq. (A.5) |
| $\theta_2(.)$ | Jump function expression defined by Eq. (A.7) |
| $\mu$ | Mean value |
| $\pi_{vA}(a)$ | Probability density for starting age amongst radiation cancer victims, both those living at the start of the exposure and those born into it ($y^{-1}$) |
| $\sigma$ | Standard deviation |
| $\sigma_Y$ | Standard deviation for age at death for a radiation cancer victim (y) |
| $\tau$ | Period from starting age to age at death for a radiation cancer victim, $y-a$ (y) |
| $\tau_i$ | Half-life for radioactive species, $i$ (y) |
| $\tau_{max}(a)$ | Maximum period between starting age and age at death for a radiation cancer victim, $\alpha_0 - a$ (y) |
| $\psi_0(.)$ | Expression given by Eq. (24) (y) |
| $\psi_1(.)$ | Expression given by Eq. (A.3) ($y^2$) |
| $\psi_1^{(T_R-b)}$ | $\psi_1$ function where the radiation exposure lasts $T_R - b$ years rather than $T_R$ years ($y^2$) |
| $\omega_1$ | Time after point radiation exposure when death from a radiation cancer first becomes possible (y) |
| $\omega_2$ | Time after point radiation exposure after which no more radiation cancer deaths will occur ($y^2$) |
| $\omega_3$ | $\omega_1 + T_R$ (y) |
| $\omega_4$ | $\omega_2 + T_R$ (y) |
| $\Omega$ | $\omega_2 - \omega_1$ (y) |

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